Journal of Southeast University (English Edition) Vol.20 No.1 Mar. 2004 ISSN 1003—7985

$L(2, 1)$-labeling problem on distance graphs

Tao Fangyun Gu Guohua

(Department of Mathematics, Southeast University, Nanjing 210096, China)

Abstract: $L(2, 1)$-labeling number, $\lambda(G(\mathbf{Z}, D))$, of distance graph $G(\mathbf{Z}, D)$ is studied. For general finite distance set D, it is shown that $2|D|+2 \leqslant \lambda(G(\mathbf{Z}, D)) \leqslant |D|^2+3|D|$. Furthermore, $\lambda(G(\mathbf{Z}, D)) \leqslant 8$ when D consists of two prime positive odd integers is proved. Finally, a new concept to study the upper bounds of $\lambda(G)$ for some special D is introduced. For these sets, the upper bound is improved to 7.

Key words: $L(2, 1)$-labeling; distance graph; channel assignment problem

The channel assignment problem is to assign a channel (nonnegative integer) to each radio transmitter so that interfering transmitters are assigned channels whose separation is not in a set of disallowed separations. Hale[1] first formulated this problem as a graph coloring problem. In 1988, Roberts (in a private communication to Griggs) introduced a variation of the channel assignment problem, where "close" transmitters must receive different channels and "very close" transmitters must receive channels that are at least two channels apart. This problem was formulated as a graph labeling problem by Griggs, et al.[2] To formulate the problem in graphs, the transmitters are represented by the vertices of a graph; two vertices are "very close" if they are adjacent in the graph and "close" if they are of distance two in the graph. More precisely, given a graph G with vertex set V and edge set E, for any $u, v \in V$, let $d_G(u, v)$ denote the distance between u and v in G. An $L(2, 1)$-labeling f is an integer assignment $f: V \to \{0, 1, 2, \cdots\}$ such that $|f(u)-f(v)| \geqslant 2$ if $d_G(u, v)=1$ and $|f(u)-f(v)| \geqslant 1$ if $d_G(u, v)=2$. Elements of the image of f are called labels. A k-$L(2, 1)$-labeling is an $L(2, 1)$-labeling such that no label is greater than k. The $L(2, 1)$-labeling number of G, denoted by $\lambda(G)$, is the smallest number k such that G has a k-$L(2, 1)$-labeling. We shall assume with no loss of generality that the minimum label of $L(2, 1)$-labelings of G is 0.

Griggs, et al.[2] and Yeh[3] determined the exact values of $\lambda(P_n)$, $\lambda(C_n)$ and $\lambda(W_n)$, where P_n is a path on n vertices, C_n is a cycle of n vertices, and W_n is an n-wheel obtained from C_n by adding a new vertex adjacent to all vertices in C_n. For a tree T with maximum degree $\Delta \geqslant 1$, Griggs, et al.[2] proved that $\lambda(T)$ is either $\Delta+1$ or $\Delta+2$. For a general graph G with maximum degree Δ, they proved that $\lambda(G) \leqslant \Delta^2+2\Delta$. This upper bound was improved to $\Delta^2+2\Delta-3$ when G is 3-connected and $\lambda(G) \leqslant \Delta^2$ when G is of diameter two. Griggs, et al. conjectured that $\lambda(G) \leqslant \Delta^2$ in general. Moreover, they proved that the $L(2, 1)$-labeling problem is NP-complete for general graphs. Chang, et al.[4] improved the upper bound $\Delta^2+2\Delta$ to $\Delta^2+\Delta$. They presented a polynomial time algorithm to determine $\lambda(T)$ of a tree T.

To study Griggs and Yeh's conjecture, the class of chordal graphs is considered[5]. It showed that $\lambda(G) \leqslant (\Delta+3)^2/4$ for any chordal graph G. For a unit interval graph G, which is a very special chordal graph, Sakai showed that $2\chi(G)-2 \leqslant \lambda(G) \leqslant \chi(G)$, where $\chi(G)$ denotes the chromatic number of G.

Aside from the graphs mentioned above, people have studied many other classes of graphs. These include complete k-partite graphs, n-cube Q_n, cographs, bipartite graphs, outerplanar and planar graphs, Cartesian product of complete graphs, Cartesian product of a cycle and a path, power paths, t-tree, etc.(see Refs.[2−4, 6−10]).

Suppose that D is a subset of all positive integers. The integer distance graph (or simply distance graph) $G(\mathbf{Z}, D)$ with distance set D is the graph with a vertex set $\mathbf{Z}$ ($\mathbf{Z}$ is the set of all integers), and two vertices u and v are adjacent if and only if $|u-v| \in D$. Integer distance graphs were introduced and studied by Eggleton, et al[11]. They were motivated by the famous plane coloring problem: What is the minimum number of colors necessary to color the points of the Euclidean plane such that pairs of points of unit distance are colored differently.

In this paper, we study the $L(2, 1)$-labeling number $\lambda(G(\mathbf{Z}, D))$. Furthermore, there have not

Received 2003-06-27.

Foundation item: The National Natural Science Foundation of China (No.10171013).

Biographies: Tao Fangyun (1979—), female, graduate; Gu Guohua (corresponding author), male, professor.

been any reports about $L(2,1)$-labeling on distance graphs so far. For general distance sets, we prove that $\lambda(G(\mathbf{Z},D)) \leqslant |D|^2+3|D|$ when $|D|$ is finite. For some special sets D, we also give better upper bounds of $\lambda(G(\mathbf{Z},D))$.

1 General Distance Sets

In this section, we focus on the general distance sets D, where $|D|$ is finite. We shall present both upper and lower bounds of $\lambda(G(\mathbf{Z},D))$.

We introduce some notions first. Suppose $D=\{a_1,a_2,\cdots,a_k\}$, $0<a_1<a_2<\cdots<a_k$, let $D^2=\{2a_i \mid 1\leqslant i\leqslant k\}\cup\{a_j\pm a_i \mid 1\leqslant i<j\leqslant k \text{ and } i\neq j\}$. It is easy to see that $|D^2|\leqslant k^2$. We let $[0,k]$ denote the set $\{0,1,\cdots,k\}$.

Lemma 1[2] Let G be a graph with maximum degree $\Delta\geqslant 2$. If G has three vertices of degree Δ such that one such vertex is adjacent to the other two, then $\lambda(G)\geqslant\Delta+2$.

Theorem 1 $2|D|+2\leqslant\lambda(G(\mathbf{Z},D))\leqslant|D|^2+3|D|$.

Proof The lower bound is obviously true by lemma 1, so it suffices to show that $\lambda(G(\mathbf{Z},D))\leqslant|D|^2+3|D|$. We define a labeling f of $G(\mathbf{Z},D)$ recursively as follows. First, $f(0)=0$. When $f(j)$ is defined for $-i\leqslant j\leqslant i$, let $f(i+1)=\min\{t: |t-x|\geqslant 2$ for all $x\in A$ and $t\notin A'\}$, where $A=\{f(j): -i\leqslant j\leqslant i$ and $i+1-j\in D\}$ and $A'=\{f(j): -i\leqslant j\leqslant i$ and $i+1-j\in D^2\}$. Then let $f(-i-1)=\min\{t: |t-x|\geqslant 2$ for all $x\in B$ and $t\notin B'\}$, where $B=\{f(j): -i\leqslant j\leqslant i+1$ and $j+i+1\in D\}$ and $B'=\{f(j): -i\leqslant j\leqslant i+1$ and $j+i+1\in D^2\}$. f is clearly a proper $L(2,1)$-labeling of $G(\mathbf{Z},D)$. Each vertex $i+1$ is adjacent to at most $|D|$ vertices in $[-i,i]$, and there are at most $|D^2|$ vertices in $[-i,i]$, which are distance 2 away from $(i+1)$. So when it is time to label $i+1$, there are at most $3|D|+|D^2|\leqslant 3|D|+|D|^2$ numbers to be avoided. There is some label in $[0,3|D|+|D|^2]$ available for $i+1$. Similarly, there is some label in $[0,|D|^2+3|D|]$ available for $-i-1$. So f is a $(|D|^2+3|D|)$-$L(2,1)$-labeling of $G(\mathbf{Z},D)$.

Corollary 1 For $|D|=1$, $\lambda(G(\mathbf{Z},D))=4$.

Remark Ref.[2] proved that $\lambda(P_n)=4$, for $n\geqslant 5$, where P_n denotes a path on n vertices. Since when $|D|=1$, each component of $G(\mathbf{Z},D)$ is isomorphic to a path on infinite vertices, corollary 1 is equivalent to the well-known result above. Moreover, it is worth pointing out that the lower bound in theorem 1 is sharp. In fact, for each positive integer k, we set $D=\{1,2,\cdots,k\}$, then $G(\mathbf{Z},D)$ is an example where this lower bound is attainable since $\lambda(G,\{1,2,\cdots,k\})=2k+2$ (see Ref.[8]). On the other hand, the upper bound may not be sharp when $|D|\neq 1$. For some special D, we may improve the upper bound. For instance, we consider the case $D=\{1,3,5,\cdots,2k-1\}$. $G(\mathbf{Z},D)$ is a subgraph of $G(\mathbf{Z},D')$ where $D'=\{1,2,3,\cdots,2k-1\}$, so $\lambda(G(\mathbf{Z},D))\leqslant\lambda(G(\mathbf{Z},D'))=2(2k-1)+2=4k$, that is to say $\lambda(G(\mathbf{Z},D))\leqslant 4|D|<|D|^2+3|D|$ when $|D|\neq 1$. The upper bound in theorem 1 can be reduced.

2 Two-Element Distance Sets

In this section, we consider the case when $|D|=2$ and try to improve the upper bound for some special D.

For the case where D is a set of positive integers with $g=\gcd(D)$, each component of $G(\mathbf{Z},D)$ is isomorphic to $G(\mathbf{Z},D')$, where $D'=\{d': gd'\in D\}$. So when we study $\lambda(G(\mathbf{Z},D))$, we may assume that $\gcd(D)=1$. For the case of $|D|=2$, where $D=\{a,b\}$, $a<b$, and a, b are relatively prime, so either they are both odd or they are of opposite parity. In the following we discuss the former case.

Theorem 2 If $D=\{a,b\}$, $a<b$, and a, b are relatively prime positive odd integers, then $6\leqslant\lambda(G(\mathbf{Z},D))\leqslant 8$.

Proof $\lambda(G(\mathbf{Z},D))\geqslant 6$ follows from theorem 1. Next we show that $\lambda(G(\mathbf{Z},D))\leqslant 8$. Since $\gcd(a,b)=1$ and both a and b are odd, $G(\mathbf{Z},D)$ contains no odd cycles. This implies that $G(\mathbf{Z},D)$ is a bipartite graph. Assume that A and B are two partition sets. Noting that G is 4-regular. We can get an $L(2,1)$-labeling for G by using the numbers in $[0,3]$ to label the vertices in A, and the numbers in $[5,8]$ to label B. This is feasible since there exist no five vertices in A (or B), such that any two of them are of distance two. To the contrary, suppose that there are five vertices in A(or B), say u, v, w, x, y, such that any two of them are distance two away from each other. Without loss of generality, we may assume $u<v<w<x<y$, which implies that $0<v-u<w-u<x-u<y-u$. So either of the following equality constraints holds:

$$\left.\begin{aligned} y-u&=2b\\ x-u&=b+a\\ w-u&=2a\\ v-u&=b-a \end{aligned}\right\} \qquad (1)$$

$$\left.\begin{array}{l} y-u=2b \\ x-u=b+a \\ w-u=b-a \\ v-u=2a \end{array}\right\} \quad (2)$$

From (1), we can deduce $y-w=2b-2a$ and $w-v=3a-b$. Since $a<b$, $2b-2a=2a$ or $b+a$, which means $b=2a$ or $3a$. So $w-v=a$ or 0, contradicting that w and v are of distance two. Similarly, a contradiction can be reached from (2). Hence the theorem holds.

In the following we will consider the case where a, b are of opposite parity. This case is more complicated. And we only get better upper bounds for some special D. First we introduce some new conceptions.

An $L(2,1)$-labeling f: $\mathbf{Z}\to\{0,1,2,\cdots\}$ is called periodic with period p if $f(v)=f(v+p)$ for all $v\in\mathbf{Z}$. We denote a p-periodic $L(2,1)$-labeling by f_p.

An $L(2,1)$-labeling f_p: $\mathbf{Z}\to\{0,1,2,\cdots\}$ is called d-consistent ($d\in\mathbf{N}$, where $\mathbf{N}$ is the set of positive integers) if $|f(v)-f(v+d)|\geqslant 2$ and $|f(v)-f(v+2d)|\geqslant 1$ for all $v\in\mathbf{Z}$.

An $L(2,1)$-labeling f_p: $\mathbf{Z}\to\{0,1,2,\cdots\}$ is called $\{d_1,d_2\}$-consistent if for all $v\in\mathbf{Z}$ it satisfies

$$\begin{cases} |f_p(v)-f_p(v+d_1)|\geqslant 2 \\ |f_p(v)-f_p(v+d_2)|\geqslant 2 \end{cases}$$

and

$$\begin{cases} |f_p(v)-f_p(v+2d_1)|\geqslant 1 \\ |f_p(v)-f_p(v+2d_2)|\geqslant 1 \\ |f_p(v)-f_p(v+d_1-d_2)|\geqslant 1 \\ |f_p(v)-f_p(v+d_1+d_2)|\geqslant 1 \end{cases}$$

By definition, we can prove the following:

Proposition 1 If f_p is a periodic d-consistent $L(2,1)$-labeling of $G(\mathbf{Z},D)$ with period $p>d$, then f_p is also $(np\pm d)$-consistent for all $n\in\mathbf{N}$.

Proof Since f_p is d-consistent, we have

$$|f_p(v)-f_p(v+d)|\geqslant 2$$

and

$$|f_p(v)-f_p(v+2d)|\geqslant 1$$

then

$$|f_p(v)-f_p(v+np\pm d)| = |f_p(v)-f_p(v\pm d)|\geqslant 2$$
$$|f_p(v)-f_p[v+2(np\pm d)]| = |f_p(v)-f_p(v\pm 2d)|\geqslant 1$$

which completes the proof.

Proposition 2 If f_p is a periodic $\{d_1,d_2\}$-consistent $L(2,1)$-labeling of $G(\mathbf{Z},D)$ with period $p>d_i$, $i=1,2$, then f_p is also $\{np\pm d_1, mp\pm d_2\}$-consistent for all n, $m\in\mathbf{N}$.

Proof Since f_p is $\{d_1,d_2\}$-consistent, we have

$$\begin{cases} |f_p(v)-f_p(v+d_1)|\geqslant 2 \\ |f_p(v)-f_p(v+d_2)|\geqslant 2 \end{cases}$$

and

$$\begin{cases} |f_p(v)-f_p(v+d_1-d_2)|\geqslant 1 \\ |f_p(v)-f_p(v+d_1+d_2)|\geqslant 1 \\ |f_p(v)-f_p(v+2d_1)|\geqslant 1 \\ |f_p(v)-f_p(v+2d_2)|\geqslant 1 \end{cases}$$

then

$$|f_p(v)-f_p(v+np\pm d_1)| = |f_p(v)-f_p(v\pm d_1)|\geqslant 2$$
$$|f_p(v)-f_p(v+mp\pm d_2)| = |f_p(v)-f_p(v\pm d_2)|\geqslant 2$$
$$|f_p(v)-f_p(v+2(np\pm d_1))| = |f_p(v)-f_p(v\pm 2d_1)|\geqslant 1$$
$$|f_p(v)-f_p(v+2(mp\pm d_2))| = |f_p(v)-f_p(v\pm 2d_2)|\geqslant 1$$
$$|f_p(v)-f_p[v+(np\pm d_1)-(mp\pm d_2)]| = |f_p(v)-f_p[v\pm(d_1\pm d_2)]|\geqslant 1$$
$$|f_p(v)-f_p[v+(np\pm d_1)+(mp\pm d_2)]| = |f_p(v)-f_p[v\pm(d_1\pm d_2)]|\geqslant 1$$

which completes the proof.

Remark We can also define $\{d_1,d_2,\cdots,d_n\}$-consistent in the same way and draw a conclusion similar to the propositions above. Since we focus on the case when $|D|=2$, the details are omitted here.

In the following we consider distance sets $D=\{x, x+s\}$ with $s=1,3,5$. In our discussion, a famous theorem of Frobenius is necessary, so we state it first.

Lemma 2[12] Let a and b be two positive integers such that the greatest common divisor of a and b is 1. If t is an integer such that $t>ab-a-b$ then the equation $t=na+mb$ has at least one solution with n and m nonnegative integers.

Theorem 3 If $D=\{x, x+s\}$, then $6\leqslant\lambda(D)\leqslant 7$ if one of the following cases occurs:

1) $s=1$ and $x>38$;

2) $s=3$ and $x>39$;

3) $s=5$ and $x>38$.

Proof Since $|D|=2$, then $\lambda(D)\geqslant 2\cdot 2+2=6$ follows from theorem 1. In the following we show that the upper bound is true by providing a 7-$L(2,1)$-labeling in all cases.

Case 1 The periodic $L(2,1)$-labelings $P_7=0123456$ and $P_8=01234567$ are obviously $\{2,3\}$-consistent. Since $\gcd(7,8)=1$ the equation $t=7n+8m$ has a nonnegative solution (n,m) for all $t>7\cdot 8-7-8=41$ by lemma 2.

For such a solution (n,m) we define a periodic 7-$L(2,1)$-labeling of the form

$$P_t = P_7^n P_8^m = \underbrace{P_7\cdots P_7}_{n}\underbrace{P_8\cdots P_8}_{m}$$

It is straightforward to verify that $P_7P_8 = 012345601234567$ and $P_8P_7 = 012345670123456$ are also $\{2, 3\}$-consistent, which implies P_t is $\{t \pm 2, t \pm 3\}$-consistent by proposition 2. If we set $t = x + 3$ then there exists for all $x > 38$ a 7-$L(2, 1)$-labeling of the form $P_t = P_{x+3}$ which is $\{x, x+1\}$-consistent.

Case 2 We use the analogous arguments above with $P_7 = 0246135$ and $P_8 = 02461357$ which are $\{1, 2\}$-consistent. And we set $t = x + 2$ to construct a $\{x, x+3\}$-consistent labeling.

Case 3 We use the same proof as in case 1.

3 Further Research

This paper first proves that $\lambda(G(\mathbf{Z}, D)) \leqslant |D|^2 + 3|D|$ when $|D|$ is finite. We concentrate on the case when $|D| = 2$. We have shown that the upper bound of $\lambda(G(\mathbf{Z}, D))$ is 8 when D contains two odd integers. For some special sets D, we improve the bound to 7. Moreover, we expect to determine the exact values of $\lambda(G(\mathbf{Z}, D))$ for these sets. We are also working on the case of $D = \{a, b\}$ where a and b are of opposite parity, we conjecture that the upper bound of $\lambda(G(\mathbf{Z}, D))$ is 8.

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距离图 $L(2, 1)$标号着色问题

陶昉昀 顾国华

(东南大学数学系, 南京 210096)

摘要: 研究了距离图 $G(\mathbf{Z}, D)$的 $L(2, 1)$-标号着色数 $\lambda(G(\mathbf{Z}, D))$. 对一般的有限距离集 D, 证明了 $2|D| + 2 \leqslant \lambda(G(\mathbf{Z}, D)) \leqslant |D|^2 + 3|D|$. 此外, 当 D 由 2 个互素正奇数构成时, 有$\lambda(G(\mathbf{Z}, D)) \leqslant 8$ 的结论.最后引入了一个新的概念对一些特殊距离图的 $\lambda(G)$ 上界进行了研究, 对于这些距离图, $\lambda(G)$ 的上界可以改进到 7.

关键词: $L(2, 1)$标号着色; 距离图; 频道分配问题

中图分类号: O157.5