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A Discrete-Time Stochastic Traffic Assignment Model*

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Abstract: A discrete-time stochastic traffic assignment model is proposed. The model provides a discrete-time description of the variations of flows on a road network during a day or a peak period. The congestion effect at links and link junctions are taken into account. The first-in-first-out principle is enforced on all links at all periods of the day. A stochastic user equilibrium assignment is achieved when the tripmaker is unable to find better travel alternatives. A computational procedure is also presented.

Key words: stochastic user equilibrium, traffic assignment, discrete-time traffic assignment

As an important aspect of the theoretic research of intelligent transportation systems, the dynamic traffic assignment models have been developed during last two decades. According to the methodology, there are four sorts of dynamic assignment models: computer simulation approach, mathematical programming approach, optimal control theoretic approach and variation inequality approach. The existing dynamic assignment models have developed from models, which can only handle simple network forms and restrict the trip decision to departure time choice, to models that can handle both departure time choice and route choice of general networks[1,2]. The dynamic assignment models can be divided into two classes: the discrete-time models and the continuous-time models.

The early developed dynamic assignment models can only solve the problems of a single O-D pair network connected by parallel routes[1]. Some models can solve the departure time choice and route choice of general networks. These models consider that the traffic condition within a link is assumed to be homogeneous[2]. Clearly, this assumption does not conform to the reality of urban traffic condition. This paper proposed a discrete-time dynamic stochastic assignment model, which provides a discrete-time description of the variations of traffic flow on a highway network during a day or a peak period. The traffic flow on links in the model is assumed as two parts: the free flow and a queue. Travelers are assumed to try to minimize their total time cost by selecting proper trip time and route. The first-in-first-out principle is enforced on all links . The congestion effect on links and on link junctions are taken into account. A demand adjustment mechanism is derived from a dynamic Markovian model. The state of the dynamic stochastic user equilibrium is achieved when no traveler believes that he can increase his total utility of travel by unilaterally changing route and departure time. Since the traffic conditions are described well conforming to the practical conditions, the model developed in this paper can model the congestion and incident affection on traffic condition more precisely.

For the convenience of study, notations are given as follows:

$G(N,A)$ is the road network with N node and A directed link;

N is the set of network nodes;

A is the set of directed link;

R is the set of origin node, $R \in N$;

S is the set of destination node, $S \in N$;

r is the number of the origin node, $r \in R$;

s is the number of the destination node, $s \in S$;

P is the set of O-D pair;

p is the number of an O-D pair, $p \in P$;

K_p is the set of routes connecting the O-D pair p, $p \in P$;

k is the number of a route of O-D pair p, $k \in K_p$;

L_{kp} is equal to $\{i_1, i_2, \cdots, i_x\}$, which is an order link set of route k of O-D pair p;

$i_1, i_2, \cdots, i_x$ are link numbers of route k of O-D pair p;

$O_k(i_x)$ is the link order number of route k of O-D pair p, $O_k(i_1) = 1, O_k(i_2) = 2, \cdots, O_k(i_x) = x$, i_1 is connected to the origin r, and i_x is connected to the destination s;

A_i^+ is the set of links leaving node i;

A_i^- is the set of links entering node i;

j_a is the head node of link a, $a = \{i, j_a\} \in A$;

i_a is the tail node of link a, $a = \{i_a, j\} \in A$;

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τ_a is the free flow time spending on going through link a (in periods), $a \in A$;

T is the total number of time intervals within the study time;

Δt is a time interval;

t is the number of time interval $t \in T$, for the convenience of expression and calculation, the time of $t\Delta t$ is represented by time t;

T_a^t is the average time spending on going through link a if link a is entered at time t;

T_{kpi}^t is the time spending on traveling from link i entered at time t to the end of link x along route k of O-D pair p;

T_{kp}^t is the average travel time spending on going through route k of O-D pair p when the first link of route k is entered at time t;

Q_p is the total number of trips between O-D pair p;

Q_p^t is the total number of vehicles in interval t which leave the origin of O-D pair p at time t;

Q_{kp}^t is the total number of vehicles in interval t which leaves the origin of O-D pair p at time t along the route k, $\sum_{k \subset K_p} Q_{kp}^t = Q_p^t$, $\forall_p \in P, t \in T$;

PR_{kp}^t is the probability that a driver, traveling along O-D pair p, will depart at time t and select route k;

T_a^t is the time required to go through link a entered at time t, $t \in T$;

v_a^t is the number of vehicles entering link a at time t;

x_{ap}^t is the number of vehicles belonging to O-D pair p entering link a at time t, $\sum_{p \subset P} x_{ap}^t = v_a^t, \forall a \in A$, $t \in T$;

x_{apk}^t is the number of vehicles belonging to O-D pair p traveling along route k entering link a at time t.

1 Travel Time Mode

Given a time varying traffic assignment, we can load the traffic flows to the network. Supposing a vehicle enters link a at time t, after spending τ_a time periods, the vehicle reaches the end of link a at time $t'(t' > t)$ and joins an existing queue in which there are $\bar{w}_a^{t'}$ vehicles. The number of vehicles in the queue at time t' is $w_a^{t'} = \bar{w}_a^{t'} + 1$.

The number of vehicles leaving the queue at time t' is determined by an exit function. The exit function of link a at time t' is the function of $w_a^{t'}$ and other factors, denoted by $u_a^{t'}$, i.e.

$$u_a^{t'} = f_a(w_a^{t'}, \cdots) \qquad \forall a \in A,\ t' \in T \qquad (1)$$

The function $f_a(\cdot)$ is assumed to be continuous with respect to its arguments and satisfies $f_a(w_a^{t'}, \cdots) < w_a^{t'}$.

Since the vehicles in the queue at time t' may enter the link at different periods of time, the FIFO principle should be enforced as follows:

$$u_a^{tt'} = \min\left\{ w_a^{tt'}, \max\left[0, u_a^{t'} - \sum_{t'' \in T | t'' < t} u_a^{t''t'}\right]\right\}, \quad \forall a \in A,\ t' \in T,\ t \in T \qquad (2)$$

where $u_a^{tt'}$ is the number of vehicles having entered the link a at time t, existing at time t'. $w_a^{tt'}$ is the number of vehicles in the exit queue at time t', having entered the link a at time t.

Suppose the traffic flows of different O-D pair are fully mixed on each link a, the traffic flow of different O-D pair traveling along different routes exiting the queue is then in proportion to the number of the vehicles of different O-D pair traveling along different routes in the queue, i.e.

$$u_{apk}^{tt'} = \begin{cases} 0 & w_a^{tt'} = 0 \\ u_a^{tt'}(w_{apk}^{tt'}/w_a^{tt'}) & \text{otherwise} \end{cases} \quad \forall a \in A,\ p \in P,\ k \in K_p,\ t \in T,\ t' \in T,\ t < t' \qquad (3)$$

where $u_{apk}^{tt'}$ is the number of vehicles of O-D the pair p traveling along route k exiting the queue at time t' having entered the link a at time t; $w_{apk}^{tt'}$ is the number of the O-D pair p traveling along route k in the exit queue at time t' having entered the link a at time t.

The number of vehicles of the O-D pair p along route k exiting link a in period t' can be deduced from Eq.(3).

$$u_{apk}^{t'} = \sum_{t \in T | t < t'} u_{apk}^{tt'} \qquad \forall a \in A,\ p \in P,\ k \in K_p,\ t' \in T \qquad (4)$$

The number of the vehicles of the O-D pair p along route k entering link a at time t must satisfy:

$$x_{apk}^t = \begin{cases} \sum_{a' \in A_{ia}^-} u_{a'pk}^t + Q_{kp}^t \cdot \delta_{apk} & i_a \in k \\ \sum_{a' \in A_{ia}^-} u_{a'pk}^t & \text{otherwise} \end{cases} \qquad (5)$$

where $\delta_{apk} = \begin{cases} 0 & \text{if } a \in k \\ 1 & \text{otherwise} \end{cases} \quad \forall a \in A,\ p \in P,\ k \in K_p$.

The number of vehicles in the queue of each link can be determined by the number of vehicles entering the link and the number of vehicles exiting the link. The free flow time is assumed to be τ_a. If τ_a is an integer, then all vehicles will reach the exit queue at time $t' = t + \tau_a$. If τ_a is a real number, it is assumed that the vehicle entrance time on the link is uniformly spread out over the period. Then a proportion $[\mathrm{int}(\tau_a) + 1 - \tau_a]$ of vehicles will reach the end of the link at time $t' = t + \mathrm{int}(\tau_a)$ and the remaining vehicles will do so at time $t' = t + \tau'_a + 1$ (int denotes the integer function and let $\tau'_a = \mathrm{int}(\tau_a)$). It can be got that

$$w_{apk}^{tt'} = \begin{cases} w_{apk}^{t(t'-1)} - u_{apk}^{t(t'-1)} & \text{if } t' > t + \tau'_a + 1 \\ w_{apk}^{t(t'-1)} - u_{apk}^{t(t'-1)} + (\tau_a - \tau'_a) x_{apk}^t & \text{if } t' = t + \tau'_a + 1 \\ (\tau_a - \tau'_a + 1) x_{apk}^t & \text{if } t' = t + \tau'_a \\ 0 & \text{otherwise} \end{cases} \quad (6)$$

$$w_a^{tt'} = \sum_{p \in P} \sum_{k \in K_p} \delta_{apk} w_{apk}^{tt'} \qquad \forall a \in A,\ t, t' \in T \quad (7)$$

$$w_a^{t'} = \sum_{t \in T | t < t'} w_a^{tt'} \qquad \forall a \in A,\ t, t' \in T \quad (8)$$

After the above analysis, we study the expected travel time going through a link which is entered at different time. The probabilities of exiting link a, which is entered at time t and t' must be firstly computed.

$$v_a^t = \sum_{p \in P} \sum_{k \in K_p} x_{apk}^t \qquad \forall a \in A,\ t \in T \quad (9)$$

If $v_a^t > 0$, then $PB_a^{tt'} = u_a^{tt'} / v_a^t$, $\forall a \in A$, $t, t' \in T$, $t < t'$, where $PB_a^{tt'}$ is the probability of exiting link a, which is entered at time t and t'.

If $v_a^t = 0$, the probability can be calculated according to four subcases depending on the length of the exit queue at the two consecutive periods $t + \tau'_a$ and $t + \tau'_a + 1$.

If $w_a^{(t+\tau'_a)} = 0, w_a^{(t+\tau'_a+1)} = 0$,

$$PB_a^{tt'} = \begin{cases} \tau'_a - \tau_a + 1 & \text{if } t' = t + \tau'_a \\ \tau_a - \tau'_a & \text{if } t' = t + \tau'_a + 1 \\ 0 & \text{otherwise} \end{cases}$$

If $w_a^{(t+\tau'_a)} > 0, w_a^{(t+\tau'_a+1)} = 0$,

$$PB_a^{tt'} = \begin{cases} \tau'_a - \tau_a + 1 & \text{if } t' = t^* \\ \tau_a - \tau'_a & \text{if } t' = t + \tau'_a + 1 \\ 0 & \text{otherwise} \end{cases}$$

If $w_a^{(t+\tau'_a)} = 0, w_a^{(t+\tau'_a+1)} > 0$,

$$PB_a^{tt'} = \begin{cases} \tau'_a - \tau_a + 1 & \text{if } t' = t + \tau'_a \\ \tau_a - \tau'_a & \text{if } t' = t^{**} \\ 0 & \text{otherwise} \end{cases}$$

If $w_a^{(t+\tau'_a)} > 0, w_a^{(t+\tau'_a+1)} > 0$,

$$PB_a^{tt'} = \begin{cases} \tau'_a - \tau_a + 1 & \text{if } t' = t^* \\ \tau_a - \tau'_a & \text{if } t' = t^{**} \\ 0 & \text{otherwise} \end{cases}$$

where t^*, t^{**} are the periods of time when the last vehicle among the $w_a^{(t+\tau'_a)}$ vehicles and the last vehicle among the $w_a^{(t+\tau'_a+1)}$ vehicles exit the queue respectively, i.e.

$$t^* = \max\{\bar{t} \in T | u_a^{t''t} > 0, t'' < t\}$$

$$t^{**} = \max\{\bar{t} \in T | u_a^{t''t} > 0, t'' \leqslant t\}$$

Hence, we can get the average travel time going through link a which is entered at time t.

$$T_a^t = \sum_{t' \in T | t' > t} (t - t') PB_a^{tt'} \qquad \forall a \in A,\ t \in T \quad (10)$$

The average travel time going through the last link of route p is computed as

$$T_{apx}^t = \sum_{t' \in T | t' > t} (t - t') PB_{i_x}^{tt'} \qquad \forall i_x \in A,\ t', t \in T,\ k \in K_p,\ p \in P$$

The travel time going through route p is calculated as follows:

$$T_{kpi}^t = \sum_{t' \in T | t' > t} PB_{i_i}^{tt'} ((t - t') + T_{kpi}^{t'}),\ j = i - 1, \forall i \in O_k(i_x), i_x \in A,\ a \in A,\ t', t \in T,\ k \in K_p, p \in P \quad (11)$$

$$T_{kp}^t = T_{kp1}^t \quad (12)$$

2 The Demand Model

Travelers are assumed to make proper decisions in the two dimensions of choice (departure time and route) in order to maximize their utility of travel. While an individual is assumed to first decision on what time to depart and then, condition on his choice, which route to follow. Thus, the probability that a driver will depart at time t and select route k is the production of the probability of departing at time t and the probability of selecting route k given at departure at time t. The equation is as follows:

$$PR_{kp}^i = \frac{\exp(\mu_r V_{kp}^t)}{\sum_{k \in K_p} \exp(\mu_r V_{kp}^t)} \frac{\exp(\mu_1 V_p^{*t})}{\sum_{k \in K_p} \exp(\mu_1 V_p^{*t})} \quad (13)$$

where V_{kp}^t is the measured utility experienced by a driver belonging to O-D pair p departing at time t and selecting route k; μ_1, μ_r are the scale parameters associated with the upper level of decision (i.e. departure time choice)and the lower level of decision (i.e. route choice) respectively, $\mu_1/\mu_r \leqslant 1$; V_p^{*t} is a composite variable which expresses the expected maximum utility from the choice of the alternative feasible routes at time t, and is defined as

$$V_p^{*t} = \frac{1}{\mu_r} \ln \sum_{k \in K_p} e^{\mu_r V_{kp}^t}$$

Let $[t_p - D_p,\ t_p + D_p]$, where $D_p > 0$, be the desired time period of arrival at the destination of a driver belonging to O-D pair p. The main sources of disutility that influence travelers' choices are travel time and schedule delay. Following the form used in existing dynamic assignment models, the disutility is assumed to be in proportion to travel time , early arrival time, schedule delay. The proportion coefficients are α, β and $\beta\gamma\gamma$, respectively. Thus, the utility function is expressed as

$$V_{kp}^t = \alpha T_{kp}^t + \beta D_p \mid \theta_{kp}^t \mid - \beta \theta_{kp}^t (t_p - t + T_{kp}^t) \quad (14)$$

where $\theta_{kp}^t = \begin{cases} 1 & \text{if } t \leqslant t_{kp}^{11} \\ 0 & \text{if } t_{kp}^{11} < t < t_{kp}^{\prime 22} \\ -\gamma & \text{if } t \geqslant t_{kp}^{22} \end{cases}$

t_{kp}^1 and t_{kp}^2 are the earliest and latest possible departure time which a traveler following route k can select. t_{kp}^{11} and t_{kp}^{22} are the integer values just smaller than t_{kp}^1

and t_{kp}^2 respectively. To guarantee a traveler to reach the destination on time, t_{kp}^1 and t_{kp}^2 must satisfy the following equations

$$t_{kp}^1 = t_p - D + T_{kp}^{t_{kp}^{11}}, t_{kp}^2 = t_p + D + T_{kp}^{t_{kp}^{22}}$$

After the utility function being defined, the probability PR_{kp}^t can be calculated. The value of Q_{kp}^t can then be defined as follows:

$$Q_{kp}^t = Q_p PR_{kp}^t \tag{15}$$

Although the equation has a concise form, the complexity of the relationships involved in the formulation of both the demand and the travel time model do not allow the derivation of analytical solutions. The equation must be solved by an iterative algorithm.

3 The Demand Adjustment Model

The dynamic demand adjustment model should represent the interaction between the transportation system's characteristics and individuals' decisions as it is directed by their own criteria of choice, and describe the evolution of the time dependent departure time rate and travel time distributions over time. For that, an extra notation is added to indicate the day variable while the setting of the system remains the same as before, e.g. V_{kp}^{td}, Q_{kp}^{td}. In the same time, it is assumed that the user of a transportation network continuously modify their trip decisions based on the information they acquired from recent trips. Thus, a traveler will either change his current trip choice searching for a better option, or remain at his current decision state when he makes trip next time.

Suppose that there is a fraction of $F_{mk}^{t'td}\Delta d$ of individuals who shift from a departure time t' to a departure time t and switch from route m to route k during the time interval $[d, d+\Delta d]$. The rate of the uncertain individual number, following route k and departing at time t , can then be expressed as the difference per unit of time between the number of individuals of shifting to t and switch to k and the number of individuals of shifting from t and/or switch from k. Let $\mathrm{d}t \to 0$, then this rate of change can be expressed as

$$\frac{\partial Q_{kp}^{td}}{\partial d} = \sum_{(t',m)\neq(t,k)} \sum_{t'} \sum_{m} Q_{mp}^{t'd} F_{mk}^{t'td} - Q_{kp}^{td} \sum_{(t',m)\neq(t,k)} \sum_{t'} \sum_{m} F_{mk}^{t'td} \tag{16}$$

Supposing the utility of a new decision state is assumed to be independent of the attributes of the current state, it implies that there is a constant transition rate out of the current decision state. Thus, the probability that a tripmaker decides to review his current trip choice is constant, F_1. Let the probability that a tripmaker will select route k and departure time t be $F_{k|c}^{td}$. The transition rate from the decision state (t', m) to the state (t, k) can be expressed as

$$F_{mk}^{t'td} = F_1 F_{k|c}^{td} \tag{17}$$

The travelers who will review their current trip decisions are assumed to be classified I into two categories: ① A traveler I belongs to the first category may alter his current trip decisions in both or at least in one of his dimensions of choice. ② An individual from the second category may switch to another route k remaining the same departure time t'. If the ratio of the number of the first category travelers to the number of the second category travelers is assumed be constant. The probability that a randomly selected traveler belongs to the first category is assumed to be a constant value F_2. The probability that he belongs to the second category is equal to $(1 - F_2)$.

The probability that a traveler I who will select a route k and a departure time t is given by the following nested logit

$$FF_{k|c1}^{td} = \frac{\exp(\mu_{\mathrm{r}} V_{kp}^{td'})}{\sum_{k\in K_p} \exp(\mu_{\mathrm{r}} V_{kp}^{td'})} \frac{\exp(\mu_{\mathrm{r}} \overset{*}{V}_{kp}{}^{td'})}{\sum_{t'\in T} \exp(\mu_{\mathrm{r}} \overset{*}{V}_{kp}{}^{t'd'})} \tag{18}$$

The probability that a traveler II who will select a route k is given by the following multinomial logit

$$FF_{k|c2}^{td} = \frac{\exp(\mu_{\mathrm{r}} V_{kp}^{t'd'})}{\sum_{k\in K_p} \exp(\mu_{\mathrm{r}} V_{kp}^{td'})} \tag{19}$$

To determine the utilities, the travelers are required to have perfect traffic information at time $d' = (d + \Delta d)$. However, a traveler cannot obtain any such information. Therefore, the variables of utility in above two equations can only be expressed as a function of w instead of d'.

Since the trip distribution varies with time, the set of reasonable route is changing depending on the time. This variability of the set of reasonable paths gives rise to the definition of two new sets of routes for each departure time t. The first set includes the routes which constitute a reasonable choice for a departure during the interval t both at time d and d', denoted by $DK_p^{td}, DK_p^{td} = K_p^{td} \cap K_p^{td'}$. The second set includes the routes which are considered as reasonable choices for a departure during the interval t at time d but not at d' denoted by $KK_p^{td}, KK_p^{td} = K_p^{td} \sim K_p^{td'} = \{k \mid k \in k_p^{td} \wedge k \notin K_p^{td'}\}$. A tripmaker first estimates the traffic patterns that will take place at time d', and then defines the set $K_p^{td'}$. If a traveler has selected a route $k \notin DK_p^{td}$ will definitely have to review his current trip choice. Thus, for the group of travelers who have selected route $k \in KK_p^{td}$ the value of F_1 will be equal to 1.

Given the values of F_1 and F_2, the number of vehicles departing during the interval t at time $(d+1)$ can be deduced with Eq.(16) and Eq.(17) as follows:

$$\begin{aligned} Q_{kp}^{td+1} = & Q_{kp}^{td} - F_1 Q_{kp}^{td} \\ & + F_1 \Big[\sum_{t' \in T} \sum_{m \in DK_p^{t'd}} Q_{mp}^{t'd} F_2 FF_{k|c1}^{td} \\ & + \sum_{m \in DK_p^{t'd}} Q_{mp}^{td} (1 - F_2) FF_{k|c2}^{td} \Big] \\ & + \sum_{t' \in T} \sum_{m \in KK_p^{t'd}} Q_{mp}^{t'd} F_2 FF_{k|c1}^{td} \\ & + \sum_{m \in KK_p^{t'd}} Q_{mp}^{td} (1 - F_2) FF_{k|c2}^{td} \end{aligned} \tag{20}$$

4 The Computation Procedure

Step 1 Give the initial condition such as the transportation network characteristics, the set of reasonable routes K_p, the number of trips for each O-D pair, the number of trips of each O-D pair for each route and departure time;

Step 2 Calculating the travel time of each link which is entered at different time with the traffic volumes and the network characteristics;

Step 3 Determine the reasonable routes for the next iteration with the values of travel time;

Step 4 Calculate the probabilities and then the value of Q_{kp}^{t};

Step 5 If the value of Q_{kp}^{t} meet the error requirement, the iteration ends. Otherwise, repeat step 2 to step 4.

5 Conclusion

This paper proposes a discrete-time stochastic traffic assignment model based on the assumption which provides a better representation of the actual traffic condition. The demand adjust model is based on the principle of dynamic stochastic user equilibrium. The model can give better description influence of congestion and incidents on the traffic condition. The further work is to implement the model in an actual network.

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离散化随机交通分配模型研究

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摘　要　提出了一种离散化交通分配模型.模型以离散形式描述一天或高峰期内路网交通流的变化,并考虑了拥挤效应和先入先出原则,利用多层随机概率模型模拟出行者出行选择,实现随机用户平衡分配.最后给出了计算方法.

关键词　随机用户平衡，交通分配,离散时间交通分配

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