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New Methods to Solve Fuzzy Shortest Path Problems*

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Abstract: This paper discusses the problem of finding a shortest path from a fixed origin s to a specified node t in a network with arcs represented as typical triangular fuzzy numbers (TFN). Because of the characteristic of TFNs, the length of any path p from s to t, which equals the extended sum of all arcs belonging to p, is also TFN. Therefore, the fuzzy shortest path problem (FSPP) becomes to select the smallest among all those TFNs corresponding to different paths from s to t (specifically, the smallest TFN represents the shortest path). Based on Adamo's method for ranking fuzzy number, the pessimistic method and its extensions — optimistic method and λ-combination method, are presented, and the FSPP is finally converted into the crisp shortest path problems.

Key words: triangular fuzzy number, fuzzy shortest path, ranking function

The classic shortest path problem (SPP) is concerned with finding the path with minimum length between two specified nodes in a network. It is an important problem because of its numerous applications and generalizations in transportation networks. Over the past several years, a lot of attention has been paid to the SPP and its applications. There are many classic algorithms based on dynamic programming procedures such as Dijkstra algorithm and Floyd algorithm to deal with it. This problem is generally regarded as one of the most fundamental problems in non-fuzzy networks[1-4].

The fuzzy shortest path problem was first analyzed by Dubois and Prade. However, the major drawback to this classical fuzzy path problem is the lack of interpretation[1]. That is, a fuzzy shortest path length can be found, but it may not correspond to an actual path in the network. Realizing the defect, C.M. Klein presented several models and developed associated algorithms for them. In fact, there are many other papers discussing FSPP, such as Refs.[5 – 7].

Let $G(V, E)$ be a simple network with node set V and arc set E. Associated with each arc $(i, j) \in E$, there is a non-negative arc length l_{ij}. Let R be the set of all paths from the origin node 1 to the destination node N. If $p \in R$, the length of p is then defined as in classical (nonfuzzy) network theory:

$$\operatorname{len}(p) = \sum_{(i,j)\in p} l_{ij}$$

Thus, the SPP (nonfuzzy) can be mathematically stated as follows:

$$p^* = \arg\min_{p\in R} \sum_{(i,j)\in p} l_{ij} \tag{1}$$

When arc lengths are fuzzy numbers (we use $\tilde{l}_{ij}$ instead of l_{ij} as arc length), Dubois and Prade discussed the solution of the problem through the use of extended sum, and extended min and max, and a fuzzy shortest path length can be found. However, a path with that length may not exist since the extended min of several fuzzy numbers may not be one of those numbers. Therefore, the solution is trivial to decision-maker[1].

In this paper, we deal with FTNs commonly as representative fuzzy numbers. The length of p is then defined as:

$$\tilde{}\operatorname{len}(p) = \sum_{(i,j)\in p} \tilde{l}_{ij}$$

According to extended sum, the length of any path p in R, $\tilde{}\operatorname{len}(p)$, is FTN also.

If there are altogether M paths in R (p_i, $i = 1, 2, \cdots, M$), and based on some approach of ranking fuzzy numbers, it has $\tilde{}\operatorname{len}(p_1) \geqslant \tilde{}\operatorname{len}(p_2) \geqslant \cdots \geqslant \tilde{}\operatorname{len}(p_{M-1}) \geqslant \tilde{}\operatorname{len}(p_M)$, then we define p_M is the fuzzy shortest path under this ranking approach.

There are many classic approaches to ranking fuzzy numbers[7-10]. Apparently, different ranking methods will give different results. Based on Adamo's approach[8], we present three ranking methods named pessimistic method, optimal method and λ-combination method respectively. All the three methods converted the FSPP into the crisp shortest path problems.

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1 Preliminaries

Let Γ be a set of non-negative triangular fuzzy numbers. $A \in \Gamma$ will be identified by its characteristic function

$$\mu_A(x) = \begin{cases} 0 & 0 \leqslant x \leqslant a \\ \dfrac{x-a}{b-a} & a < x \leqslant b \\ \dfrac{x-c}{b-c} & b < x \leqslant c \\ 0 & x > c \end{cases} \quad 0 \leqslant a < b < c$$

$$\mu_A(x) = \begin{cases} 0 & x = a \\ 1 & x \neq a \end{cases} \quad 0 \leqslant a = b = c$$

or, more simply, by the triplet (a, b, c). (a, b, c) denotes a crisp number if $a = b = c$.

Thus

$$\Gamma = \{(a,b,c) \mid a < b < c \text{ or } a = b = c,\ a, b, c \text{ are non-negative real numbers}\}$$

According to the extension principle, the sum of two TFNs: (a_1, b_1, c_1) and (a_2, b_2, c_2) is

$$(a_1, b_1, c_1) + (a_2, b_2, c_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2) \tag{2}$$

There are many articles about ranking or comparing fuzzy numbers[6,8-10]. In this paper, we proposed three ranking methods on the base of the Adamo's method whose robustness under particular hypotheses (overlapping/non-overlapping, convex/non-convex, normal/non-normal fuzzy sets) is tested[11].

Suppose that we have n normal convex fuzzy subsets $\tilde{u}_i$ ($\tilde{u}_i = (a_i, b_i, c_i), i = 1, 2, \cdots, n$) on real line E, a simple method to order them is to define a ranking function F mapping each fuzzy set into E, where a natural order exists. This approach has been applied by Adamo[8].

Hence

$$F: \wp(E) \to E$$

where $\wp(E)$ is the set of fuzzy subsets of E. F is such that

$F(\tilde{u}_i) < F(\tilde{u}_j)$ implies $\tilde{u}_i < \tilde{u}_j$

$F(\tilde{u}_i) = F(\tilde{u}_j)$ implies $\tilde{u}_i = \tilde{u}_j$

$F(\tilde{u}_i) > F(\tilde{u}_j)$ implies $\tilde{u}_i > \tilde{u}_j$

Adamo[11] uses the concept of α-level set to obtain a α-preference index which is given by

$$F_\alpha(\tilde{u}_i) = \max\{z \mid \mu_{\tilde{u}_i}(z) \geqslant \alpha\} \tag{3}$$

for a given threshold $\alpha \in [0,1]$. Fig.1 shows an example for $\alpha = 0.7$, and it's easy to see

$$(1,4,9) <_{0.7} (3,5,8) \qquad (\text{Using } (3))$$

Proposition 1 If $\tilde{u}_i = (a_i, b_i, c_i)$, $\tilde{u}_j = (a_j, b_j, c_j)$ then using (3), we have

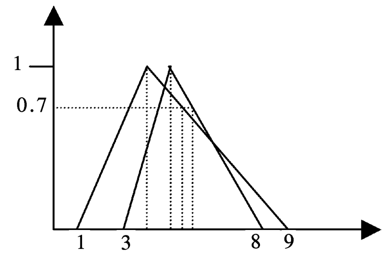

Fig.1 0.7 preference index

$$\tilde{u}_i \leqslant_\alpha \tilde{u}_j \Leftrightarrow (1-\alpha)c_i + \alpha b_i \leqslant (1-\alpha)c_j + \alpha b_j$$

Proof From (3) we know

$$F_\alpha(\tilde{u}_i) = \max\{z \mid \mu_{\tilde{u}_i}(z) \geqslant \alpha\} = (1-\alpha)c_i + \alpha b_i$$

$$F_\alpha(\tilde{u}_j) = \max\{z \mid \mu_{\tilde{u}_j}(z) \geqslant \alpha\} = (1-\alpha)c_j + \alpha b_j$$

Apparently,

$$\tilde{u}_i \leqslant_\alpha \tilde{u}_j \Leftrightarrow (1-\alpha)c_i + \alpha b_i \leqslant (1-\alpha)c_j + \alpha b_j$$

Further, for $\alpha = 0.5$,

$$\tilde{u}_i \leqslant_{0.5} \tilde{u}_j \Leftrightarrow c_i + b_i \leqslant c_j + b_j \tag{4}$$

When $\alpha = 0.5$, the preference relation "$\leqslant_{0.5}$" is equivalent to "$\leqslant_o$" proposed by G. Faccinetti[6].

2 The Pessimistic Method

Now, let's come back to the network $G(N, A)$.

Let the triplet (a_{ij}, b_{ij}, c_{ij}) denote $\tilde{l}_{ij}$, the fuzzy length of any arc (i, j), that is

$$\tilde{l}_{ij} = (a_{ij}, b_{ij}, c_{ij})$$

And let $\widetilde{\mathrm{Len}}(p)$ denote the fuzzy length of p ($p \in R$), we define

$$\widetilde{\mathrm{Len}}(p) = \sum_{(i,j)\in p} \tilde{l}_{ij}$$

Thus

$$\widetilde{\mathrm{Len}}(p) = \sum_{(i,j)\in p} (a_{ij}, b_{ij}, c_{ij}) = \left(\sum_{(i,j)\in p} a_{ij}, \sum_{(i,j)\in p} b_{ij}, \sum_{(i,j)\in p} c_{ij}\right) \tag{5}$$

The FSPP is

$$p^* = \arg\min_{p\in R} F_\alpha(\widetilde{\mathrm{Len}}(p)) \tag{6}$$

A naive approach to exactly determine the fuzzy shortest path is to enumerate all paths, compute the $\widetilde{\mathrm{Len}}(p)$ and compare them for all $p \in R$ by Adamo's ranking method. However, since the number of paths grows exponentially with the size of the network, this approach becomes intractable for even a moderate size network. In fact, this problem essentially is the crisp shortest path problem.

According to (5) and proposition 1, (6) is equivalent to

$$p^* = \arg\min_{p\in R} (1-\alpha)\sum_{(i,j)\in p} c_{ij} + \alpha \sum_{(i,j)\in p} b_{ij}$$

$$= \arg\min_{p \in R} \sum_{(i,j) \in p} [(1-\alpha)c_{ij} + \alpha b_{ij}] \quad (7)$$

Eq.(7) is a typical shortest path problem that can be easily solved by Dijkstra algorithm.

The pessimistic method considers only the right hand of a TFN. Specifically, the decision-maker compares paths using bigger value that could be obtained from a fuzzy number for the shortest path problem. Therefore, this method will be used by a conservative decision-maker.

3 Optimistic Method

In contrast to the method above (see (3)), we propose the second method to rank TFNs. It will be used by an optimistic decision-maker.

We define another α-preference index which is given by

$$F'_\alpha(\tilde{u_i}) = \min\{z \mid \mu_{\tilde{u_i}}(z) \geqslant \alpha\} \quad (8)$$

Similar to proposition 1, it has

Proposition 2 If $\tilde{u_i} = (a_i, b_i, c_i)$, $\tilde{u_j} = (a_j, b_j, c_j)$ then using (8), we have

$$\tilde{u_i} \leqslant_\alpha \tilde{u_j} \Leftrightarrow (1-\alpha)a_i + \alpha b_i \leqslant (1-\alpha)a_j + \alpha b_j$$

Proof It leaves to readers.

Fig. 1 shows an example for $\alpha = 0.7$. Because $F'_\alpha(\tilde{u_i}) = 3.1$, $F'_\alpha(\tilde{u_j}) = 4.4$, it's easy to see

$$(1,4,9) <_{0.7} (3,5,8) \quad \text{(Using (8))}$$

By using (8), the FSPP becomes (see (6) for comparison)

$$p^* = \arg\min_{p \in R} F'_\alpha(\tilde{\ }\mathrm{Len}(p)) \quad (9)$$

Accordingly

$$p^* = \arg\min_{p \in R} \sum_{(i,j) \in p} [(1-\alpha)a_{ij} + \alpha b_{ij}] \quad (10)$$

Eq. (10) is called the optimistic method for we only compare paths using smaller value that could be obtained from the left hand of a TFN for the shortest path problem, and it will be used by an optimistic decision-maker.

4 λ-Combination Method

In this section, we will give a general method of choice that could be used when the decision-maker prefers an intermediate situation between the pessimistic and optimistic points of view.

We define the third α-preference index as follows

$$F''_\alpha(\tilde{u_i}) = \lambda F_\alpha(\tilde{u_i}) + (1-\lambda)F'_\alpha(\tilde{u_i}), \quad \lambda \in [0,1] \quad (11)$$

It's easy to see that

① If $\lambda = 1$ then $F''_\alpha(\tilde{u_i}) = F_\alpha(\tilde{u_i})$;

② If $\lambda = 0$ then $F''_\alpha(\tilde{u_i}) = F_\alpha'(\tilde{u_i})$.

From Fig.1, we can see that for $\alpha = 0.7$ and $\lambda = 0.5$.

$$F''_\alpha(\tilde{u_i}) = 0.5(5.5 + 3.1) = 4.3$$

$$F''_\alpha(\tilde{u_j}) = 0.5(5.9 + 4.4) = 5.15$$

Thus

$$(1,4,9) <_{0.7} (3,5,8) \quad \text{(Using (11))}$$

Further if $\tilde{u} = (a, b, c)$, then

$$F''_\alpha(\tilde{u}) = \lambda[(1-\alpha)c + \alpha b] + (1-\lambda)[(1-\alpha)a + \alpha b]$$

The FSPP becomes

$$p^* = \arg\min_{p \in R} F''_\alpha(\tilde{\ }\mathrm{Len}(p)) \quad (12)$$

Accordingly

$$p^* = \arg\min_{p \in R} \sum_{(i,j) \in p} \{\lambda[(1-\alpha)c_{ij} + \alpha b_{ij}] + (1-\lambda)[(1-\alpha)a_{ij} + \alpha b_{ij}]\} \quad (13)$$

5 Comparison with DP Recursion Method

In this paper, we use fuzzy extended sum of corresponding arc length to evaluate the fuzzy length of a path, the path with minimum fuzzy length based on a ranking criterion is regarded as fuzzy shortest path. In order to distinguish other relevant definition, we call it sum fuzzy shortest path (SFSP). It's interesting to see all the three methods have some special characteristics.

Let us define an operator "$\overset{\sigma}{\min}$", and we will present another method based on DP recursion.

Suppose $\{x_1, x_2, \cdots, x_m\} \subset \Gamma$.

If according to a ranking criterion function σ, $x_{i_1} \geqslant x_{i_2} \geqslant \cdots \geqslant x_{i_m}$, then we define

$$\overset{\sigma}{\min}\{x_1, x_2, \cdots, x_m\} = x_{i_m}$$

The fuzzy DP recursion can be denoted as follows[1]

$$\bar{f}(N) = \{0/1\}$$

$$\bar{f}(i) = \widetilde{\min_{i<j}}\{\bar{c}_{ij} + \bar{f}(j) \mid (i,j) \in E\}$$

where $\bar{c}_{ij}$ is the fuzzy arc length, x/y represents the element x and its membership grade y, and $\bar{f}(i)$ is the fuzzy shortest path distance from i to N.

In order to distinguish the SFSP, the shortest path derived from DP recursion is called DPFSP. If we use the operator "$\overset{\sigma}{\min}$" instead of the extended min operator "$\widetilde{\min}$", it has

$$\bar{f}(i) = \overset{\sigma}{\min_{i<j}}\{\bar{c}_{ij} + \bar{f}(j) \mid (i,j) \in E\}$$

Different from $\widetilde{\min}$, $\overset{\sigma}{\min}$ will correspond to an actual path in the network. Therefore, according to the modified DP methods, not only a fuzzy shortest path length can be found, but also it corresponds to an actual path in the network.

Apparently, we can derive the following theorem.

Theorem 1 Based on the ranking criterion function $F_\alpha, F_\alpha^{'}$ and $F_\alpha^{''}$, the DPFSP is SFSP, and conversely.

From the discussion above, we see, F_α, $F_\alpha^{'}$ and $F_\alpha^{''}$ have some characteristic. That is linearity.

Definition 1 A criterion function F has linearity if and only if $\forall A \in \Gamma$, $B \in \Gamma$

$$F(A + B) = F(A) + F(B)$$

According to definition 1, $F_\alpha, F_\alpha^{'}$ and $F_\alpha^{''}$ apparently have linearity.

Theorem 2 If a criterion function F has linearity, then based on the criterion F, the DPFSP is SFSP.

6 Conclusion

There are many methods to deal with the fuzzy shortest path problem. The classic shortest path problem proposed by Dubois and Prade is lack of interpretation. That is, although a fuzzy shortest path length can be found, it may not correspond to an actual path in a network. However, from another point of view, we define the fuzzy shortest path corresponding to an actual path (See (6), (9) or (12)). Based on Adamo's method for ranking fuzzy number, the pessimistic method, optimistic method and λ-combination method are presented. Owing to the linearity of these ranking criterion functions, the problem can be easily solved by classic shortest path methods.

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模糊最短路问题的新方法

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摘 要 本文讨论三角形模糊网络中节点 s 到终点 t 的最短路问题.根据三角形模糊数(TFN)的性质可知,连结节点 s 和 t 的任何路 p 的长度(p 所经过路径的长度的扩展和)也是三角形模糊数.因此,模糊网络最短路问题本质上就是 TFN 的选择比较问题,即在连结 s 和 t 的所有路中选择长度(TFN)最小的一个.根据 Adamo 的模糊数悲观排序方法,以及它的扩展——乐观排序方法和 λ-组合排序方法,模糊网络最短路问题最终可以转化为确定网络的最短路问题.

关键词 三角形模糊数，模糊最短路，排序函数

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