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Strong Uniqueness of Best Approximations from RS-sets in Banach Spaces*

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Abstract: The problem of strong uniqueness of best approximation from an RS-set in a Banach space is considered. For a fixed RS-set G and an element $x \in X$, we proved that the best approximation g^* to x from G is strongly unique.

Key words: best approximation, RS-set, strong uniqueness

Let X be a real Banach space and G be a closed subset in X. For a point $x \in X$, an element $g^* \in G$ is called a best approximation to x from G if it satisfies that

$$\|x - g^*\| \leqslant \|x - g\| \qquad \forall g \in G$$

The set of all best approximation to x from G is denoted by $P_G(x)$, that is

$$P_G(x) = \{g^* \in G: \|x - g^*\| = d(x, G)\}$$

where $d(x, G) = \inf_{g \in G} \|x - g\|$.

For a convex subset G in X, there has been systematic theory on characterization and uniqueness of best approximation from G in the spaces with some convexity[1-3]. In order to obtain the uniqueness result in arbitrary spaces, we need to make further restriction on the set G. Motivated by the work of Rozema and Simith[4], Amir[5] introduced the concept of RS-sets and gave the uniqueness for restricted Chebyshev center of any compact set with respect to an RS-set. Recently, there are several papers concerned with the uniqueness of best approximation from an RS-set[6-8]. As is well known, strong uniqueness is an important property in approximation theory and applied in continuity of the best approximation operator as well as convergence analysis of the algorithm. However, because of the difficulty, there is almost no research concerned with the strong uniqueness of the best approximation from RS-sets. The purpose of the present paper is to give the strong uniqueness of the best approximation from RS-sets.

1 Preliminaries

Let B^* denote the closed unit ball of the dual X^* and $\mathrm{ext}B^*$ the set of all extreme points of B^*.

Definition 1 An n-dimensional subspace G of X is called an interpolating subspace if no nontrivial linear combination of n linearly independent extreme points of the ball B^* annihilates G.

Definition 2 Let $\{y_1, y_2, \cdots, y_n\}$ be n linearly independent elements of X. We call the set

$$G = \left\{g = \sum_{i=1}^{n} c_i y_i :\ c_i \in J_i\right\}$$

an RS-set if each J_i is a subset of the reals R of one of the following types:

(I) The whole of R;

(II) A nontrivial proper closed (bounded or unbounded) interval of R;

(III) A singleton.

And in addition every subset of $\{y_1, y_2, \cdots, y_n\}$ consisting of all y_i with J_i of type (I) and some y_i with J_i of type (II) spans an interpolating subspace.

For convenience, define

$$\alpha_i = \inf J_i,\ \beta_i = \sup J_i \qquad i = 1, \cdots, n$$

Then

$$\begin{array}{ll} \alpha_i = -\infty, \beta_i = +\infty & \text{if } J_i \text{ is of type (I)} \\ -\infty \leqslant \alpha_i < \beta_i \leqslant +\infty & \text{if } J_i \text{ is of type (II)} \\ \alpha_i = \beta_i \neq +\infty & \text{if } J_i \text{ is of type (III)} \end{array}$$

For $g^* = \sum_{i=1}^{n} c_i^* y_i \in G$, set

$$I_0 = \{i: \alpha_i = \beta_i\}$$

$$J_+(g^*) = \{i: c_i^* = \alpha_i\} \setminus I_0$$

$$J_-(g^*) = \{i: c_i^* = \beta_i\} \setminus I_0$$

$$J(g^*) = J_+(g^*) \cup J_-(g^*)$$

and $\sigma_i(g^*) = 1$, if $i \in J+(g^*)$; $\sigma_i(g^*) = -1$, if $i \in J_-(g^*)$.

Finally, let

$$P = \left\{g = \sum_{i=1}^{n} c_i y_i \in G:\ c_i = 0, \forall i \in I_0\right\}$$

The following characterization theorem, which proof is similar to that of theorem 2 in [7], plays a key

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role for strong uniqueness of the best approximation from RS-sets in the next section.

Theorem 1 Suppose that G is an RS-set in X and $x \in X$. Then $g^* \in P_G(x)$ if and only if there exist $A(x-g^*) = \{a_1^*, \cdots, a_l^*\} \subset E(x-g^*)$, $B(g^*) = \{j_{l+1}, \cdots, j_m\} \subset J(g^*)$ and m positive numbers $\lambda_1, \cdots, \lambda_m$ with $\sum_{i=1}^m \lambda_i = 1$ ($1 \leqslant m \leqslant \dim P + 1$) such that

$$\sum_{i=1}^l \lambda_i \langle a_i^*, g\rangle + \sum_{i=l+1}^m \lambda_i \sigma_{j_i}(g^*) c_{j_i} = 0,$$
$$\forall g = \sum_{i=1}^n c_i y_i \in P \tag{1}$$

where

$$E(x-g^*) = \{a^* \in \text{ext}B^*: \langle a^*, x-g^*\rangle = \|x-g^*\|\}$$

2 Strong Uniqueness of the Best Approximation

Lemma 1 Suppose G is an RS-set in X, $x \in X \setminus G$ and $g^* \in P_G(x)$. Let $A(x-g^*) = \{a_1^*, \cdots, a_l^*\} \subset E(x-g^*)$ and $B(g^*) = \{j_{l+1}, \cdots, j_m\} \subset J(g^*)$ be given by theorem 1. Then there is at least $N = \dim P - m + l$ linearly independent elements in $A(x-g^*)$.

Proof Let m positive numbers $\lambda_1, \cdots, \lambda_m$ satisfy $\sum_{i=1}^m \lambda_i = 1$ ($1 \leqslant m \leqslant \dim P + 1$) with $A(x-g^*)$ and $B(g^*)$ such that (1) holds. Set

$$Q = \left\{g = \sum_{j=1}^n c_j y_j \in P: c_{j_i} = 0, i = l+1, \cdots, m\right\}$$

Then Q is an interpolating subspace of dimension $N = \dim P - m + l$. With no loss of generality, we may assume that $a_1^*, \cdots, a_{l'}^*$ are linearly independent and (1) can be rewritten into

$$\sum_{i=1}^{l'} \lambda_i' \langle a_i^*, g\rangle + \sum_{i=l+1}^m \lambda_i \sigma_{j_i}(g^*) c_{j_i} = 0,$$
$$\forall g = \sum_{i=1}^n c_i y_i \in P \tag{2}$$

where all $\lambda_i' \neq 0$. In order to complete the proof, it suffices to show that $l' \geqslant N$. Suppose on the contrary that $l' < N$. Since Q is an interpolating subspace of dimension $N = \dim P - m + l$, there exists an element $g_0 \in Q \setminus \{0\}$ such that

$$\langle a_i^*, g_0\rangle = \lambda_i' \qquad i = 1, \cdots, l'$$

which implies that (2) does not hold for $g = g_0$. This yields a contradiction and completes the proof.

Lemma 2 Suppose G is an RS-set in X, $x \in X \setminus G$ and $g^* \in P_G(x)$. Then

$$\max_{a^* \in E(x-g^*)} \langle a^*, g^* - g\rangle > 0 \quad \forall g \in G \setminus \{g^*\}$$

Proof Suppose that there exists $g_0 \in G \setminus \{g^*\}$ such that

$$\max_{a^* \in E(x-g^*)} \langle a^*, g^* - g_0\rangle \leqslant 0$$

Assume that $g^* - g_0 = \sum_{i=1}^m c_i^0 y_i$. Then from theorem 1, we have

$$\sum_{i=1}^l \lambda_i \langle a_i^*, g^* - g_0\rangle + \sum_{i=l+1}^m \lambda_i \sigma_{j_i}(g^*) c_{j_i}^0 = 0$$

since $g_0 - g^* \in P$. In addition lemma 1 implies that $l \geqslant \dim P - m + l$, $\lambda_i > 0$, $i = 1, \cdots, m$. Note that $\sigma_{j_i}(g^*) c_{j_i}^0 \leqslant 0, i = l+1, \cdots, m$, it follows that

$$0 \geqslant \sum_{i=1}^l \lambda_i \langle a_i^*, g^* - g_0\rangle$$
$$= -\sum_{i=l+1}^m \lambda_i \sigma_{j_i}(g^*) c_{j_i}^0 \geqslant 0$$

so that

$$\langle a_i^*, g^* - g_0\rangle = 0 \qquad i = 1, \cdots, l$$
$$c_{j_i}^0 = 0 \qquad i = l+1, \cdots, m$$

This implies that $g^* - g_0 \in Q$ so that $g^* = g_0$, which contradicts that $g_0 \in G \setminus \{g^*\}$, and completes the proof.

Now we are ready to give the main theorem of this section.

Theorem 2 Suppose that G is an RS-set in X and $x \in X$. Then $g^* \in P_G(x)$ is strongly unique, that is, there exists a positive number $c > 0$ such that

$$\|x-g\| \geqslant \|x-g^*\| + c\|g-g^*\| \qquad \forall g \in G$$

Proof Let

$$I_+ = \{i: \alpha_i \neq -\infty\} \setminus I_0$$
$$I_- = \{i: \beta_i \neq +\infty\} \setminus I_0$$

For any $i \in I_+ \cup I_-$, define

$$H_i^+ = \left\{\sum_{j=1}^n c_j y_j \in G: c_i = \alpha_i,\right\} \qquad \text{if } i \in I_+$$
$$H_i^- = \left\{\sum_{j=1}^n c_j y_j \in G: c_i = \beta_i,\right\} \qquad \text{if } i \in I_-$$

Furthermore, let

$$\mathscr{H} = \{H_i^+; i \in I_+\} \cup \{H_i^-: i \in I_-\}$$
$$\mathscr{H}(g^*) = \{H \in \mathscr{H}: g^* \in H\}, G^0 = \bigcup_{H \notin \mathscr{H}(g^*)} H$$

Then G^0 is a closed nonempty subset of G so that $t^* = d(g^*, G^0) > 0$.

Claim 1 For any $g = \sum_{i=1}^n c_i y_i \in G \setminus \{g^*\}$, set

$$g_\lambda = \left(1 - \frac{\lambda}{\|g^* - g\|}\right) g^* + \frac{\lambda}{\|g^* - g\|} g$$
$$= \sum_{i=1}^n \left(\left(1 - \frac{\lambda}{\|g^* - g\|}\right) c_i^* + \frac{\lambda}{\|g^* - g\|} c_i\right) y_i$$

Then $g_{t^*} \in G$.

Proof Write

$$I^+ = \{i: c_i(t^*) < \alpha_i\}, I^- = \{i: c_i(t^*) > \beta_i\}$$

where

$$c_i(\lambda) = \left(1 - \frac{\lambda}{\|g^* - g\|}\right)c_i^* + \frac{\lambda}{\|g^* - g\|}c_i,$$
$$i = 1, \cdots, n$$

Suppose that $g_{t^*} \notin G$. Then, from

$$c_i(t^*) \geqslant \alpha_i \qquad \forall i \in J_+(g^*)$$
$$c_i(t^*) \leqslant \beta_i \qquad \forall i \in J_-(g^*)$$

we have that

$$I^+ \cup I^- \neq \varnothing, \ (I^+ \cup I^-) \cap J(g^*) = \varnothing$$

For any $i \in I^+ \cup I^-$, let $0 < \lambda_i^+, \lambda_i^- < t^*$ satisfy $c_i(\lambda_i^+) = \alpha_i$, $c_i(\lambda_i^-) = \beta_i$, respectively, and define $\lambda = \min\left\{\begin{matrix}\min\lambda_i^+ & \min\lambda_i^- \\ i \in I^+ & i \in I^-\end{matrix}\right\}$. It follows from the fact that $c_i(\lambda) = \alpha_i = \beta_i$ for any $i \in I_0$ and λ that $g_\lambda \in \partial G$, the boundary of G, and $J(g_\lambda) \cap (I^+ \cup I^-) \neq \varnothing$. Take $i_0 \in J(g_\lambda) \cap (I^+ \cup I^-)$. With no loss of generality, assume that $i_0 \in J_+(g_\lambda)$. Then $g^* \notin H_{i_0}^+$, $g_\lambda \in H_{i_0}^+$. Hence

$$t^* = d(g^*, G^0) \leqslant \|g^* - g_\lambda\| = \lambda < t^*$$

This is a contradiction and so $g_{t^*} \in G$, complete the proof.

Claim 2 Define

$$r(g) = \max_{a^* \in E(x-g^*)} \frac{t^*\langle a^*, g^* - g\rangle}{\|g^* - g\|}$$

Then $r = \inf_{g \in G \setminus \{g^*\}} r(g) > 0$.

Proof Suppose on the contrary that there exists a sequence $\{g_n\} \subset G \setminus \{g^*\}$ such that $r(g_n) \to 0$ as $n \to \infty$. Due to the compactness, we may assume that $\frac{t^*(g^* - g_n)}{\|g^* - g_n\|} \to \bar{g} \neq 0$. Let

$$g_n^{t^*} = \left(1 - \frac{t^*}{\|g^* - g_n\|}\right)g^* + \frac{t^*}{\|g^* - g_n\|}g_n$$

Then

$$g_n^{t^*} = g^* - \frac{t^*(g^* - g_n)}{\|g^* - g_n\|} \to g^* - \bar{g}$$

In addition, it follows from claim 1 that $g_n^{t^*} \in G$ so that $g^* - \bar{g} \in G \setminus \{g^*\}$. Observe that

$$r(g_n) = \max_{a^* \in E(x-g^*)} \langle a^*, g^* - g_n\rangle$$
$$\to \max_{a^* \in E(x-g^*)} \langle a^*, g^* - (g^* - \bar{g})\rangle$$

it follows that

$$\max_{a^* \in E(x-g^*)} \langle a^*, g^* - (g^* - \bar{g})\rangle = 0$$

which contradicts to lemma 2. This proves the claim.

Now let us return to the proof of the theorem. Note that claim 2 implies that

$$\max_{a^* \in E(x-g^*)} \langle a^*, g^* - g\rangle \geqslant rt^* \|g^* - g\| \quad \forall g \in G$$

Thus, we have that

$$\|x - g\| \geqslant \max_{a^* \in E(x-g^*)} \langle a^*, x - g^*\rangle$$
$$+ \max_{a^* \in E(x-g^*)} \langle a^*, g^* - g\rangle$$
$$\geqslant \|x - g^*\| + rt^* \|g^* - g\|$$

for all $g \in G$, that is, g^* is a strongly unique best approximation to x from the RS-set G and proves the theorem.

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Banach 空间中 RS 集最佳逼近的强唯一性

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摘　要　研究了 Banach 空间中的 RS 集的最佳逼近的强唯一性问题,对给定的 RS 集 G 及 $x \in X$,证明了 G 中对 x 的最佳逼近 g^* 的强唯一性.

关键词　最佳逼近,RS 集,强唯一性

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