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The Circular Chromatic Number of Some Special Graphs

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Abstract: The circular chromatic number of a graph is a natural generalization of the chromatic number. Circular chromatic number contains more information about the structure of a graph than chromatic number does. In this paper we obtain the circular chromatic numbers of special graphs such as C_k^t and $C_k^t - v$, and give a simple proof of the circular chromatic number of $H_{m,n}$.

Key words: circular chromatic number, graph C_k^t, graph $C_k^t - v$, graph $H_{m,n}$

Let G be a graph with vertex set V and edge set E. The chromatic number $\chi(G)$ of a graph is the least number of colors required to color the vertices of G such that adjacent vertices are assigned different colors.

The circular chromatic number of a graph, also named star chromatic number, is a natural generalization of the chromatic number[1,2].

Definition 1 Let k and d be positive integers such that $k \geqslant 2d$. A (k,d)-coloring of graph $G = (V,E)$ is a mapping $c: V \to Z_k = \{0,1,\cdots,k-1\}$, such that $d \leqslant | c(x) - c(y) | \leqslant k - d$ for any edge xy in E. The circular chromatic number $\chi_c(G)$ of graph G is defined as

$$\chi_c(G) = \inf\{k/d : G \text{ has a } (k,d)\text{-coloring}\}$$

An equivalent definition of the circular chromatic number of a graph G is given by Zhu[2]. It is easy to see that a $(k,1)$-coloring of G is just an ordinary k-coloring of G. Refs. [1,3,4] have discussed the foundermental perspectives of the circular chromatic number.

Theorem 1[1] If G is a simple graph on n vertices, then

$$\chi_c(G) = \min\{k/d : G \text{ has a } (k,d)\text{-coloring and } k \leqslant n\}$$

Theorem 2[1] For all graphs, $\chi(G) - 1 < \chi_c(G) \leqslant \chi(G)$.

Theorem 2 tells that two graphs with the same chromatic number may have different circular chromatic numbers. In some sense, $\chi_c(G)$ is a refinement of $\chi(G)$ and it contains more information about the structure of the graph G than $\chi(G)$ does.

To estimate the circular chromatic number of a graph G, Guichard[5] gives the following definition and theorem.

Definition 2 For a (k,d)-coloring $c: V \to Z_k$ of a graph G, define the digraph $D_c(G)$ which has the same vertex set as G and a directed arc from vertex u to vertex v if u is adjacent to v in G and $c(v) = c(u) + d (\mathrm{mod}\, k)$.

Theorem 3[2] $\chi_c(G) < k/d$ if and only if $D_c(G)$ is acyclic for some (k,d)-coloring c of G.

In order to understand the circular chromatic number of general graphs, it is necessary to investigate the circular chromatic number of some special classes of graphs.

Our main results are theorem 8, theorem 9 and theorem 10.

1 Circular Chromatic Number of C_k^t and $C_k^t - v$

Definition 3 Let k and d be positive integers such that $k \geqslant 2d$, G_k^d is the graph with vertices set $V = \{x_0, x_1, \cdots, x_{k-1}\}$ in which $x_i x_j$ is an edge if and only if $d \leqslant | i - j | \leqslant k - d$.

Theorem 4[2] For any positive integers k and d such that $k \geqslant 2d$, we have $\chi_c(G_k^d) = k/d$.

Ref. [2] shows that a graph G is (k,d)-colorable if and only if there exists a homomorphism from G to G_k^d. Therefore, in the study of circular chromatic numbers, graphs G_k^d play the role of the complete graphs as in study of chromatic numbers.

Definition 4 Let k, d and t be positive integers such that $k = td + i, t \geqslant 2, d \geqslant 2$ and $0 < i < t$, C_k^t

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is the graph with vertex set $V = \{x_0, x_1, \cdots, x_{k-1}\}$ in which x_ix_j is an edge if and only if $|i-j| < t$ or $|i-j| > k-t$.

Definition 5 Suppose G and H are graphs. A homomorphism of G to H is a mapping f from $V(G)$ to $V(H)$ such that $(f(u), f(v)) \in E(H)$ whenever $(u,v) \in E(G)$.

C^2_{2d+1} and C^3_{3d+i} are called odd cycles and square cycles, respectively. Comparing definition 3 and definition 4, we can find C^t_k is the complement of G^t_k. At the same time, by using function $f(i) = id(\mathrm{mod}k)$ to change vertex order of C^t_k, we get to know that C^t_k is a subgraph of G^d_k, when $(k,d) = 1$, and there exists a homomorphism from C^t_k to $G^{d'}_{k'}$ when $(k,d) = s > 1$, where $k' = k/s$, $d' = d/s$. It is clear that $G^{d'}_{k'}$ is a subgraph of G^d_k with the same circular chromatic number. It follows $\chi_c(C^t_k) \leqslant \chi_c(G^d_k) = k/d$.

In this section, we will discuss the circular chromatic number of C^t_k and $C^t_k - v$. First we introduce some results on the circular chromatic number of a graph[2-5].

Theorem 5[4,5] Let $v(G)$ and $\alpha(G)$ be the numbers of vertices of graph G and the independence number of graph G, then $\chi_c(G) \geqslant v(G)/\alpha(G)$.

Theorem 6[2] For any vertex $v \in V(G^d_k)$, we have $\chi_c(G^d_k - v) < k/d$.

Theorem 7[3] Let $(k,d) = 1$ and $\alpha d = \beta k + 1$, then for any vertex $v \in V(G^d_k)$, we have

$$\chi_c(G^d_k - v) = (k-\alpha)/(d-\beta)$$

Then we give the circular chromatic number of C^t_k and $C^t_k - v$.

Theorem 8 Let $k = td + i$, then $\chi_c(C^t_k) = t + i/d = k/d$ for any $0 < i < \min(t,d)$.

Proof We prove the theorem according to the different bounds of $\chi_c(C^t_k)$.

Claim 1 $\chi_c(C^t_k) \leqslant k/d$.

From the definition of circular chromatic number we know if there exits a (k,d)-coloring of G then $\chi_c(G) \leqslant k/d$. It means we can use the value of k/d as the upper bounds for $\chi_c(G)$. According to the definition of C^t_k, either C^t_k is a subgraph of G^d_k when $(k,d) = 1$ or exists a homomorphism from C^+_k to the subgraph of G^d_k when $(k,d) > 1$. We both have $\chi_c(C^t_k) \leqslant \chi_c(G^d_k) = k/d$.

Claim 2 $\chi_c(C^t_k) \geqslant k/d$.

Now we can consider the lower bounds of the circular chromatic number of $\chi_c(C^t_k)$. The parameters of graph C^t_k in theorem 5 are: $\alpha(C^+_k) = d$ and $v(C^t_k) = td + i$. Then we learn

$$\chi_c(C^t_k) \geqslant v(C^t_k)/\alpha(C^t_k) = (td+i)/d = k/d$$

From the result of claim 1 and claim 2, we have $\chi_c(C^t_k) = t + i/d = k/d$.

Theorem 9 For any vertex $v \in V(C^t_k)$, we have $\chi_c(C^t_k - v) = (k-1)/d = t + (i-1)/d$.

Proof Similarly to the proof of theorem 8, we can identify the upper and lower bound of $\chi_c(C^t_k - v)$.

The number of vertices of graph $C^t_k - v$ is $v(C^t_k - v) = td + i - 1$, and the independence number of graph $C^t_k - v$ is still $\alpha(C^t_k - v) = d$. According to theorem 5

$$\chi_c(C^t_k - v) \geqslant \frac{v(C^t_k - v)}{\alpha(C^t_k - v)} = \frac{td + i - 1}{d} = t + \frac{i-1}{d}$$

On the other hand, $C^t_k - v$ is the subgraph of graph C^t_{k-1}. From theorem 8, we learn

$$\chi_c(C^t_k - v) \leqslant \chi_c(C^t_{k-1}) = (k-1)/d = t + (i-1)/d$$

So it follows that $\chi_c(C^t_k - v) = (k-1)/d = t + (i-1)/d$.

2 A Simple Proof of the Circular Chromatic Number of $H_{m,n}$

Definition 6 Wheel W_n is a graph from a cycle C_n by adding a center vertex v adjacent to all vertices of C_n. Now let two edges $ab \in E(W_{2m+1})$ and $cd \in E(W_{2n+1})$ be not incident with the centers of W_{2m+1} and W_{2n+1} respectively. The Hajós sum $H_{m,n}$ of W_{2m+1} and W_{2n+1} is the graph obtained from the disjoint union of W_{2m+1} and W_{2n+1} by identifying a and c, deleting the edges ab and cd, adding an edge bd.

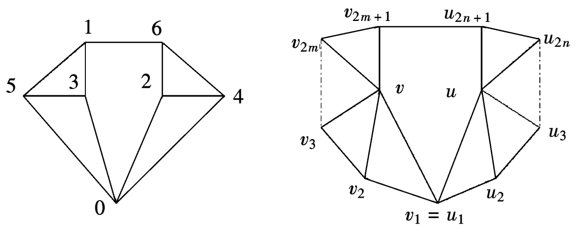
Ref. [6] gives the circular chromatic number of graph $\chi_c(H_{m,n})$. Later we give a simple proof of this result.

Theorem 10[6] For any integers $m, n \geqslant 1$, $\chi_c(H_{m,n}) = 7/2$.

Proof We prove the theorem according to the following two claims.

Claim 1 $\chi_c(H_{m,n}) \leqslant 7/2$

Fig. 1 gives a $(7,2)$-coloring of $H_{1,1}$. By mapping vertex v_{2i+1} $(i = 1, 2, \cdots, m)$ to v_1, u_{2i+1} $(i = 1, 2, \cdots, n)$ to u_1, v_{2i} $(i = 1, 2, \cdots, m)$ to v_2 and u_{2i} $(i = 1, 2, \cdots, n)$ to u_2, $H_{m,n}$ is homomorphic to $H_{1,1}$. So we have $\chi_c(H_{m,n}) \leqslant 7/2$.



(a) $H_{1,1}$ (b) $H_{m,n}$

Fig.1 Graph $H_{m,n}$

Claim 2 $\chi_c(H_{m,n}) \geqslant 7/2$.

By theorem 3 it is enough to show that for any (7, 2)-coloring of $H_{m,n}$, $D_c(H_{m,n})$ contains a directed cycle. Suppose $c: V \to Z_7$ is a (7,2)-coloring of $H_{m,n}$ such that $c(v_1) = c(u_1) = 0$, then $\{c(v_2), c(v), c(u), c(u_2)\} \subset \{2,3,4,5\}$.

If $c(v_2) = 2$ and $c(v) = 4$, then $c(v_{2i}) \in \{1, 2\}$ and $c(v_{2i+1}) \in \{0,6\}$ for any $1 \leqslant i \leqslant m$.

If $c(v_2) = 4$ and $c(v) = 2$, then $c(v_{2i}) \in \{4, 5\}$ and $c(v_{2i+1}) \in \{0,6\}$ for any $1 \leqslant i \leqslant m$.

If $c(v_2) = 2$ and $c(v) = 5$, then $c(v_{2i}) \in \{2, 3\}$ and $c(v_{2i+1}) \in \{0,1\}$ for any $1 \leqslant i \leqslant m$.

If $c(v_2) = 5$ and $c(v) = 2$, then $c(v_{2i}) \in \{4, 5\}$ and $c(v_{2i+1}) \in \{0,6\}$ for any $1 \leqslant i \leqslant m$.

If $c(v_2) = 3$ and $c(v) = 5$, then $c(v_{2i}) \in \{2, 3\}$ and $c(v_{2i+1}) \in \{0,1\}$ for any $1 \leqslant i \leqslant m$.

If $c(v_2) = 5$ and $c(v) = 3$, then $c(v_{2i}) \in \{5, 6\}$ and $c(v_{2i+1}) \in \{0,1\}$ for any $1 \leqslant i \leqslant m$.

It is to say that $c(v_{2m+1}) \in \{0,1,6\}$. Similar arguments applied on W_{2n+1}, we have $c(v_{2n+1}) \in \{0, 1,6\}$, too.

Thus we may assume, without loss of generality, $c(v_{2m+1}) = 6$ and $c(v_{2n+1}) = 1$. From above arguments, we distinct the following cases. When $c(v_2) = 2$ and $c(v) = 4$, if $c(u) = 3$ and $c(u_2) = 5$, then the cycle $v_1 \to v_2 \to v \to v_{2m+1} \to v_{2n+1} \to v_{2n} \to u \to v_1$ is a directed cycle in $D_c(H_{m,n})$; if $c(u) = 5$ and $c(u_2) = 3$ or $c(u) = 5$ and $c(u_2) = 2$, then $c(u_{2n}) = 3$ and $v_1 \to v_2 \to v \to v_{2m+1} \to v_{2n+1} \to v_{2n} \to u \to v_1$ is a directed cycle.

Similarly when $c(v_2) = 4$ and $c(v) = 2$ or $c(v_2) = 5$ and $c(v) = 2$, then $c(v_{2m}) = 4$.

So we have a directed cycle either $v_1 \to v \to v_{2m} \to v_{2m+1} \to v_{2n+1} \to u \to u_2 \to v_1$ or $v_1 \to v \to v_{2m} \to v_{2m+1} \to v_{2n+1} \to v_{2n} \to u \to v_1$.

All cases show that $D_c(H_{m,n})$ is not acyclic for any (7,2)-coloring of $H_{m,n}$ and $\chi_c(H_{m,n}) \geqslant 7/2$.

So it follows $\chi_c(H_{m,n}) = 7/2$ for any integers m, $n \geqslant 1$.

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几类特殊图的圆色数性质

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摘　要　圆色数是图的色数概念的推广.与色数相比,圆色数包含了更多有关图本身结构的信息,因而更加难以确定.本文推导了 2 类特殊图——图 C_k^t 和图 $C_k^t - v$ 的圆色数;并给出了图 $H_{m,n}$ 圆色数的一个简单证明.

关键词　圆色数,图 C_k^t,图 $C_k^t - v$,图 $H_{m,n}$

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