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Neighborhood Conditions for Claw-Free Graphs *

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Abstract: Two new sufficient conditions for hamiltonian claw-free graphs are given. Some known results become corollaries of the conclusion, the conditions of theorem are the best possible in a sense.

Key words: neighborhood, sufficient condition, hamiltonian, claw-free

We use Bondy and Murty[1] for terminology and notation not defined here and consider only simple graph. Let G be a graph of order n, if G has a Hamilton cycle, the G is called hamiltonian. A graph is called claw-free if it does not contain a copy of $K_{1,3}$ as an induced subgraph. The connectivity of G and its independence are respectively denoted by $k(G)$ and $\alpha(G)$. For $k \leqslant \alpha(G)$, let σ_k denote the minimum value of the degree-sum of any k pair-wise nonadjacent vertices. For $k = 1$, we use the usual notation $\delta(G)$ or δ if there is no confusion. The neighborhood of a vertex v is denoted by $N(v)$ and $d(v) = |N(v)|$ is the degree of the vertex v. If A, B are subgraphs of G, we define

$$N(A) = \bigcup_{v \in V(A)} N(v)$$

$$N_B(A) = N(A) \cap V(B)$$

A cycle C is called a longest cycle if there is no cycle C' such that $V(C) \subset V(C')$. If C is a cycle of a graph G, we denoted by $\vec{C}$ the cycle C with a given orientation. If $u, v \in V(C)$, then $u\vec{C}v$ denotes the consecutive vertices on $\vec{C}$ from u to v, the same vertices, in reverse order, are given by $v\overleftarrow{C}u$. We use u^+ to denote the successor of u on $\vec{C}$ and u^- to denote its predecessor, denote $u^{+2} = (u^+)^+$, $u^{-2} = (u^-)^-$. For $A \subseteq V(G)$ and a subgraph B of G, $G[A]$ and $G \setminus B$ denote the graph of G induced by A and $V(G) - V(B)$, respectively.

There are many results concerning hamiltonism and claw-free graph, the following results are known.

Theorem 1[2] A 2-connected claw-free graph with $\delta(G) \geqslant (n-2)/3$ is hamiltonian.

Theorem 2[3] Let G be a k-connected ($k \geqslant 2$) claw-free graph. If $\sigma_{k+1} \geqslant n - k$, then G is hamiltonian.

Theorem 3[4] Let G be a 2-connected claw-free graph of order $n \geqslant 14$. If for any pair of nonadjacent vertices u and v of G, $|N(u) \cup N(v)| \geqslant (2n-5)/3$, then G is hamiltonian.

Theorem 4[5] Let G be a 2-connected claw-free graph of order $n \geqslant 3$ with connectivity k. If for any pair of nonadjacent vertices u and v of G, $|N(u) \cup N(v)| > (2n - 3k + 1)/3$, then G is hamiltonian.

In fact, these results are motivated by the following conjecture of Matthews and Sumner in [6].

Conjecture[6] Let G be a 4-connected claw-free graph, then G is hamiltonian.

In this paper, we shall prove a stronger result, since it shows the existence of two vertices satisfying the given condition in any independent set is enough to guarantee the hamiltonicity. It also generalized many results above, including [1] and [5].

The principal results of the paper are as follows.

Theorem 5 Let G be a 2-connected claw-free graph of order $n \geqslant 3$, with connectivity k. If for every independent set of cardinality $k+1$, there exists $u \neq v$ in S such that

$$|N(u) \cup N(v)| \geqslant n - \delta - k$$

or $$|N(u) \cap N(v)| \geqslant n - \delta - 3k$$

then G is hamiltonian.

Theorem 6 Let G be a 2-connected claw-free graph of order $n \geqslant 3$, with connectivity k. If for every independent set of cardinality $k+1$ and for every pair of nonadjacent vertices u and v

$$d(u) + d(v) \geqslant (2n - 3k + 2)/3$$

then G is hamiltonian.

By theorem 6, consider the graph G, with connectivity $k = 2$. It suffices to show that for every pair of nonadjacent vertices u and v implies that $d(u) + d(v) \geqslant (2n-4)/3$, so the condition of theorem 6 is the best possible in a sense.

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2 Proof of Theorem 5

Let G be a 2-connected claw-free graph of order $n \geqslant 3$ with connectivity k. If G is hamiltonian, we have proved. By contradiction, we assume that G is nonhamiltonian. Let C be a longest cycle of G with a fixed orientation, B be any component of $G \setminus V(C)$. Let $N_c(B) = \{v_1, v_2, \cdots, v_m\}$, $N^- = \{v_1^-, v_2^-, \cdots, v_m^-\}$ and $N^+ = \{v_1^+, v_2^+, \cdots, v_m^+\}$. Since G is 2-connected, $k \geqslant m \geqslant 2$. Let x_j be a vertex of B and it is adjacent to v_j (for $i \neq j$, possibly $x_i = x_j$). For any $v_i, v_j \in N_c(B)$ with $i \neq j$, let v_iBv_j be a path of length at least 2 which join v_i and v_j with all internal vertices of path in B, the following claim is clear.

Claim 1 For any vertex $x \in V(B)$ and for any $i \neq j$ with $1 \leqslant i, j \leqslant m$, we have ① $N^+ \cup \{x\}$ and $N^- \cup \{x\}$ are independent sets; ② $v_j^- v_j^+ \in E(G)$; ③ $v_i^+, v_i^- \notin N(v_j)$ and $v_j^+, v_j^- \notin N(v_i)$.

For any j with $1 \leqslant j \leqslant m$, by claim 1, suppose that there exists a vertex $v_{hj} \in \{v_{j-1}^+, v_{j-1}^{+2}, \cdots, v_j^-\}$, such that $v_{hj}v_j^+ \notin E(G)$ with h_j of largest possible value (indices taken modulo m), we have the following claim.

Claim 2 For any j with $1 \leqslant j \leqslant m$, $v_{hj} \neq v_j^-$, and $v_{hj} \neq v_{j-1}^+$.

Proof Otherwise, if $v_{hj} = v_j^-$, then $v_{hj}v_j^+ \in E(G)$, by claim 1②, a contradiction. And if $v_{hj} = v_{j-1}^+$, then $v_{hj}v_{j-1}^- \in E(G)$, thus $C' = v_{hj}v_{j-1}Bv_j\overleftarrow{C} v_{hj}^+ v_j^+ \overrightarrow{C} v_{j-1}^- v_{hj}$ is a longer cycle than C, a contradiction. Let

$$H_j = \{v_{hj}, v_{hj}^+, \cdots, v_j\}$$

$$H = \{\bigcup_{t=1}^{m} H_t \setminus H_j\} \cup V(B) \cup \{v_j\}$$

Claim 3 For any j with $1 \leqslant j \leqslant m$, $N(v_{hj}) \cap H = \varnothing$.

Proof Suppose otherwise. Let $v \in N(v_{hj}) \cap H$, obviously, $v \notin V(B)$. If $v = v_r$ $(r \neq j)$, then $C' = v_{hj}v_rBv_jv_j^- \overleftarrow{C} v_{hj}^+ v_j^+ \overrightarrow{C} v_r^- v_r^+ \overrightarrow{C} v_{hj}$ is a longer cycle than C; if $v \in H_r \setminus \{v_r\}$, then $C'' = v_{hj}v\overleftarrow{C} v_j^+ v_{hj}^+ \overrightarrow{C} v_jBv_r\overleftarrow{C} v^+ v_r^+ \overrightarrow{C} v_{hj}$ is a longer cycle than C. And if $v = v_j$, then the $G[x_j, v_j, v_{hj}, v_j^+]$ is a claw, this contradicts the hypothesis of the theorem. Therefore claim 3 holds.

For any $v_i, v_j \in N_c(B)$ with $i \neq j$, without loss of generality, assume that $i < j$, denote $R(v_{hi}) = \{v \mid v \in N(v_{hi}) \setminus V(C);$ or $v^+ \in N(v_{hi}) \cap \{v_{hi}^+, v_{hi}^{+2}, \cdots, v_{hj}^-\}$; or $v^- \in N(v_{hi}) \cap \{v_{hj}^+, v_{hj}^{+2}, \cdots, v_{hi}^-\}\}$.

Claim 4 $N(v_{hj}) \cap R(v_{hi}) = \varnothing$.

Proof Suppose otherwise. Let $w \in N(v_{hj}) \cap R(v_{hi})$, if $w \in N(v_{hj}) \setminus V(C)$, it is easy to verify that there exits a cycle longer than C in G, a contradiction; if $w \in V(C)$, where $w^- \in N(v_{hi}) \cap \{v_{hj}^+, v_{hj}^{+2}, \cdots, v_{hi}^-\}$, then $C' = v_{hi}w^- \overleftarrow{C} v_j^+ v_{hj}^+ \overrightarrow{C} v_jBv_i\overleftarrow{C} v_{hi}^+ \overrightarrow{C} v_{hj}ww^+ \overrightarrow{C} v_{hi}$ is a longer cycle than C, a contradiction.

For $w^+ \in N(v_{hi}) \cap \{v_{hi}^+, v_{hi}^{+2}, \cdots, v_{hj}^-\}$, by a similar proof as the above. This is a contradiction too, so claim 4 is proved.

Consider the vertex v_{hj} and the neighborhood union $|N(v_{hj}) \cup N(x)|$. Let $F = N(v_{hj}) \cup N(x)$, assume $w \in F$. A function $f(w)$ is defined by

$$f(w) = \begin{cases} w & w \in V(G) \setminus V(C) \\ w^+ & w \in \{v_{hi}^+, v_{hi}^{+2}, \cdots, v_{hj}^{-2}\} \\ x & w = v_{hj}^- \\ w^- & w \in \{v_{hj}^+, v_{hj}^{+2}, \cdots, v_{hi}^-\} \end{cases}$$

Since C is the longest cycle in G, by claim 3 and claim 4, it is easy to check the following claim.

Claim 5 $N(v_{hi}) \cap f(F) = \varnothing$.

Assume, without loss of generality, that $S = \{v_{h1}, v_{h2}, \cdots, v_{hk}, x\}$ is a dependent set of cardinality $k + 1$, we have the following claim.

Claim 6 For any pair of vertices u and v is S, $|N(u) \cup N(v)| \leqslant n - \delta - k - 1$.

Proof By claim 3 and the define of the f, if $k = 2$, clearly $i = 1$, $j = 2$, then $\{v_{h1}, v_1, v_2\} \cap (N(v_{h1}) \cup f(F)) = \varnothing$. Thus

$$\begin{aligned} |N(v_{h2}) \cup N(x)| &= |f(F)| \\ &\leqslant n - d(v_{h1}) - |\{v_{h1}, v_1, v_2\}| \\ &= n - d(v_{h1}) - 3 \leqslant n - \delta - 3 \end{aligned}$$

And if $k \geqslant 3$, then $((\bigcup_{t \neq i,j}^{m} H_t \setminus \{v_t^-, v_t\}) \cup \{v_{hi}, v_i, v_j\}) \cap (N(v_{hi}) \cup f(F)) = \varnothing$. Thus

$$\begin{aligned} |N(v_{hj}) \cup N(x)| &= |f(F)| \leqslant n - d(v_{hi}) \\ &\quad - (|\bigcup_{t \neq i,j}^{m} H_t \setminus \{v_t^-, v_t\}| + |\{v_{hi}, v_i, v_j\}|) \\ &\leqslant n - \delta - (k - 2 + 3) = n - \delta - k - 1 \end{aligned}$$

Similarly, consider the neighborhood union $N(v_{hi}) \cup N(v_{hj})$. By claim 2 and claim 3, if $k = 2$, then

$$\begin{aligned} |N(v_{h1}) \cup N(v_{h2})| &\leqslant n - (|V(B)| \\ &\quad + |\{v_1, v_2\}|) - |\{v_{h1}, v_{h2}\}| \\ &\leqslant n - (d(x) + 1) - 2 = n - \delta - k - 1 \end{aligned}$$

And if $k \geqslant 3$, then

$$\begin{aligned} |N(v_{hi}) \cup N(v_{hj})| &\leqslant n - (|V(B)| \\ &\quad + |N_c(B)|) - |\bigcup_{t \neq i,j}^{m} H_t \setminus \{v_t\}| - |\{v_{hi}, v_{hj}\}| \\ &\leqslant n - (d(x) + 1) - 2(k - 2) - 2 \\ &= n - d(x) - 2k + 1 < n - \delta - k - 1 \end{aligned}$$

We now consider the neighborhood intersections $N(v_{hj}) \cap N(x)$ and $N(v_{hi}) \cap N(v_{hj})$, clearly, $N(v_{hj}) \cap N(x) = \varnothing$.

Claim 7 For any pair of vertices u and v in S, $|N(u) \cap N(v)| \leqslant n - \delta - 3k - 1$.

Proof Since C is the longest cycle of G, $N(v_{hi}) \cap N(v_{hj}) \subset V(C)$. By claim 2 and claim 3 and the hypothesis of theorem 5, we have $(N(v_{hi}) \cap N(v_{hj})) \cap (V(B) \cup N_c(B) \cup N^+) \cap (\bigcup_{t=1}^{m} H_t \setminus \{v_t\}) = \varnothing$. Thus

$$\begin{aligned} |N(v_{hi}) \cap N(v_{hj})| &\leqslant n - (|V(B)| \\ &+ |N_c(B)|) - (|N^+| + |\bigcup_{t=1}^{m} H_t \setminus \{v_t\}|) \\ &\leqslant n - (d(x)+1) - 3k \leqslant n - \delta - 3k - 1 \end{aligned}$$

Using claim 6 and claim 7, we can obtain a contradiction by the hypothesis of theorem 5.

The proof of theorem 5 is complete.

3 Proof of Theorem 6

We can use some arguments of theorem 5.

By the proof of claim 4, $|N(v_{hj}) \cup N(x)| \leqslant n - d(v_{hi}) - k - 1$. Since $N(v_{hj}) \cap N(x) = \varnothing$ and by the hypothesis $d(v_{hj}) + d(x) \geqslant (2n - 3k + 2)/3$, we have

$$\begin{aligned} d(v_{hi}) &\leqslant n - (d(v_{hj}) + d(x)) - k - 1 \\ &\leqslant n - (2n - 3k + 2)/3 - k - 1 \\ &= (n - 5)/3 \end{aligned}$$

Since $d(x) + d(v_{hi}) \geqslant (2n - 3k + 2)/3$

$$d(x) \geqslant (2n - 3k + 2)/3 - d(v_{hi})$$

and so

$$\begin{aligned} d(x) &\geqslant (2n - 3k + 2)/3 - (n - 5)/3 \\ &= (n - 3k + 7)/3 \end{aligned}$$

By claim 2 and claim 3, if $v_{hi}^- \in N(v_{hj})$, then $v_{hj}^- \notin N(v_{hi})$. Thus

$$\begin{aligned} d(v_{hi}) + d(v_{hj}) &\leqslant n - (|V(B)| + |N_c(B)|) \\ &- (|\bigcup_{t \neq i,j}^{m} H_t \setminus \{v_t\}| + |\{v_{hi}, v_{hj}\}|) + 1 \\ &\leqslant n - d(x) - 1 - 2(k-2) - 2 + 1 \\ &= n - d(x) - 2k + 2 \\ &\leqslant n - (n - 3k + 7)/3 - 2k + 2 \\ &= (2n - 3k - 1)/3 < (2n - 3k + 2)/3 \end{aligned}$$

this is contrary to the hypothesis of theorem 6.

Theorem 6 is proved.

Corollary Let G be a 2-connected claw-free graph of order $n \geqslant 3$ with connectivity k. If for any pair of nonadjacent vertices u and v of G,

$$|N(u) \cup N(v)| \geqslant \min\{(2n - 3k + 1)/3, \\ n - \delta - 2k + 2\}$$

then G is hamiltonian.

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无爪图的邻集条件

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摘　要　给出了关于无爪 Hamilton 图的两个新的充分条件,其结果可推出一些已知的结果,在某种意义下,条件是最好可能的.

关键词　邻集, 充分条件, 哈密尔顿图, 无爪

中图分类号　O157.5