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Slodkowski Joint Spectrum and Tensor Product

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Abstract: Slodkowski joint spectrum is similar to Taylor joint spectrum, but it has more important meaning in theory and application. In this paper we characterize Slodkowski joint spectrum and generalize some results about tensor product.

Key words: operators, joint spectrum, tensor product, Koszul complex

1 Preliminaries

Let H be a complex Hilbert space, $L(H)$ denotes the Banach algebra of all bounded linear operators on H. The space $L(H)^{(n)}$ consists of all n-tuples of bounded linear operators $T = (T_1, \cdots, T_n)$ on H. $L(H)^{(n)}_{\text{com}}$ denotes the subset of all commuting n-tuples in $L(H)^{(n)}$ (i.e., $T_iT_j = T_jT_i$, $i \neq j$), clearly $L(H)^{(n)}_{\text{com}}$ is closed in $L(H)^{(n)}$.

If $T = (T_1, \cdots, T_n) \in L(H)^{(n)}_{\text{com}}$, then $E(T,H)$ represents the Koszul complex associated with T[1,2]. From [3], we have that $E(T,H)$ is naturally isomorphic to the following complex $E'(T,H)$: $0 \to H_n \xrightarrow{D_n^{(n)}} H_{n-1} \to \cdots \to H_k \xrightarrow{D_k^{(n)}} H_{k-1} \to \cdots \xrightarrow{D_1^{(n)}} H_0 \to 0$, where $H_k = H \otimes C^{\binom{n}{k}} = H \otimes C^{\binom{n-1}{k}} \oplus H \otimes C^{\binom{n-1}{k-1}}$, for $k = 0, 1, \cdots, n$.

$$D_k^{(n)} = \begin{pmatrix} D_k^{(n-1)} & (-1)^{k+1}\operatorname{diag} T_n \\ 0 & D_{k-1}^{(n-1)} \end{pmatrix}$$

for $k = 0, 1, \cdots, n$, and

$$D_1^{(n)} = (T_1, \cdots, T_n), D_n^{(n)} = \begin{pmatrix} (-1)^{n+1} T_n \\ \vdots \\ T_1 \end{pmatrix}$$

$$D_k^{(n)} = 0$$

for all $k \leqslant 0$ or $k \geqslant n$.

Let $H_k(T,H) = \operatorname{Ker} D_k^{(n)} / \operatorname{Ran} D_{k+1}^{(n)}$ denote the k-th homological vector space. The straightforward calculation implies that

$$H_n(T,H) = \ker T = \bigcap_{i=1}^{n} \ker T_i$$

$$H_0(T,H) = H \Big/ \sum_{i=1}^{n} T_i H$$

On the other hand, let

$$\sum\nolimits_j (T,H) = \{\lambda \in C^n : H_j(T-\lambda, H) \neq 0\}$$
$$= \{\lambda \in C^n : E'(T-\lambda, H) \text{ is not exact at } j\}$$

$\sigma_{\delta,k}(T,H) = \bigcup_{j=0}^{k} \sum_j (T,H)$, $\sigma_{\pi,k}(T,H) = \bigcup_{j=n-k}^{n} \sum_j (T,H) \cup \{\lambda \in C^n : \operatorname{Ran} D_{n-k}^{(n)}(T-\lambda) \text{ is not closed}\}$, $\forall k \geqslant 0$, $\sigma_{\delta,k}, \sigma_{\pi,k}$ are called Slodkowski joint spectrum[1].

Remark 1 Slodkowski joint spectrum has the polynomial spectral mapping theorem and $\sigma_{\pi,k}(T,H) = \sigma_{\delta,k}(T,H) = \sigma_T(T)$ $(k \geqslant n)$, where $\sigma_T(T)$ is Taylor joint spectrum.

Remark 2 $\sigma_{\pi,0}(T,H) = \sigma_\pi(T)$, which is the left joint spectrum, and $\sigma_{\delta,0}(T,H) = \sigma_\delta(T)$, which is the right joint spectrum.

Remark 3 $\sigma_{\pi,k}(T,H) = \sigma_{\delta,k}(T^*, H)^*$.

2 Main Results

In this section, first we generalize a result of [3], and possess an equivalent characterization of Slodkowski joint spectrum.

Theorem 1 Some notions are as above, then

(i) $0 \notin \sigma_{\delta,k}(T,H) \Leftrightarrow D_i^{(n)^*} D_i^{(n)} + D_{i+1}^{(n)} D_{i+1}^{(n)^*}$ is invertible on $H \otimes C^{\binom{n}{i}}$, $i = 0, 1, \cdots, k$.

(ii) $0 \notin \sigma_{\pi,k}(T,H) \Leftrightarrow D_{n-i}^{(n)^*} D_{n-i}^{(n)} + D_{n-i+1}^{(n)} D_{n-i+1}^{(n)^*}$ is invertible on $H \otimes C^{\binom{n}{n-i}}$, $i = 0, 1, \cdots, k$.

Proof We will only need to show (i), for $\sigma_{\pi,k}(T,H) = \sigma_{\delta,k}(T^*, H)^*$. $0 \notin \sigma_{\delta,k}(T,H) \Rightarrow H_i(T,H) = 0$ $(i = 0, \cdots, k)$. Since $D_0^{(n)} = 0$ and $E'(T,H)$ is exact at $i = 0$, $D_1^{(n)} D_1^{(n)^*}$ is invertible. Suppose $D_i^{(n)^*} D_i^{(n)} + D_{i+1}^{(n)} D_{i+1}^{(n)^*}$ is invertible for $i \leqslant k-1$.

Now, we claim that $D_k^{(n)^*} D_k^{(n)} + D_{k+1}^{(n)} D_{k+1}^{(n)^*}$ is invertible. Using the exactness at k of $E'(T,H)$, we obtain that $H_k = \operatorname{Ker} D_k^{(n)} \oplus \operatorname{Ran} D_k^{(n)^*}$. Then, for $\forall y \in H_k$, there exist $x_0 \in \operatorname{Ker} D_k^{(n)}$ and $x_1 \in \operatorname{Ran} D_k^{(n)} = \operatorname{Ker} D_{k-1}^{(n)}$ such that $y = x_0 + D_k^{(n)^*} x_1$. Since $x_0 \in$

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$\mathrm{Ker}D_k^{(n)}$, there exists an $x_2 \in H_{k+1}$ such that $x_0 = D_{k+1}^{(n)} x_2$ and $y = D_{k+1}^{(n)} x_2 + D_k^{(n)^*} x_1$, it follows from the exactness at $i = k-1$, that (1) $y = D_{k+1}^{(n)} x_2 + D_k^{(n)^*} D_k^{(n)} y_1$, for some $y_1 \in H_k$. $\mathrm{Ran}D_{k+1}^{(n)} = \mathrm{Ran}(D_{k+1}^{(n)} D_{k+1}^{(n)^*})^{\frac{1}{2}}$ implies that (2) $D_{k+1}^{(n)} x_2 = (D_{k+1}^{(n)} D_{k+1}^{(n)^*})^{\frac{1}{2}} x_2'$ for some $x_2' \in H_k$. However, there exist $x_3 \in \mathrm{Ker}D_k^{(n)}$ and $x_4 \in H_{k-1}$ such that $x_2' = x_3 + D_k^{(n)^*} x_4$, and $x_3 \in \mathrm{Ker}D_k^{(n)}$ implies that for some $x_5 \in H_{k+1}, x_3 = D_{k+1}^{(n)} x_5 \in \mathrm{Ran}D_{k+1}^{(n)} = \mathrm{Ran}(D_{k+1}^{(n)} D_{k+1}^{(n)^*})^{\frac{1}{2}}$. Thus (3) $x_3 = (D_{k+1}^{(n)} D_{k+1}^{(n)^*})^{\frac{1}{2}} y_2$, for some $y_2 \in H_k$. From (1), (2) and (3), we have $y = D_k^{(n)} x_2 + D_k^{(n)^*} D_k^{(n)} y_1 = (D_{k+1}^{(n)} D_{k+1}^{(n)^*})^{\frac{1}{2}} x_2' + D_k^{(n)^*} D_k^{(n)} y_1 = D_{k+1}^{(n)} D_{k+1}^{(n)^*} y_2 + D_k^{(n)^*} D_k^{(n)} y_1$ (since $D_{k+1}^{(n)} D_{k+1}^{(n)^*} D_k^{(n)^*} = 0$, $(D_{k+1}^{(n)} D_{k+1}^{(n)^*})^{\frac{1}{2}} D_k^{(n)^*} = 0$), therefore, $y = D_k^{(n)^*} D_k^{(n)} y_1 + D_{k+1}^{(n)} D_{k+1}^{(n)^*} y_2$, and $y_1, y_2 \in H_k$.

From $H_k = \mathrm{Ker}D_k^{(n)} \oplus \mathrm{Ran}D_k^{(n)^*}$, we can choose that $y_1 \in \mathrm{Ran}D_k^{(n)^*}$, $y_2 \in \mathrm{Ker}D_k^{(n)}$. Hence, $\forall\, y \in H_k$, there exist $y_2 \in \mathrm{Ker}D_k^{(n)}$, $y_1 \in \mathrm{Ran}D_k^{(n)^*}$ such that $y = (D_k^{(n)^*} D_k^{(n)} + D_{k+1}^{(n)} D_{k+1}^{(n)^*})(y_1 + y_2)$, whence $D_k^{(n)^*} D_k^{(n)} + D_{k+1}^{(n)} D_{k+1}^{(n)^*}$ is invertible on H_k.

Conversely, if $D_i^{(n)^*} D_i^{(n)} + D_{i+1}^{(n)} D_{i+1}^{(n)^*}$ is invertible on H_i, $i = 0, 1, \cdots, k$, then for $x \in \mathrm{Ker}D_i^{(n)}$, $(D_i^{(n)^*} D_i^{(n)} + D_{i+1}^{(n)} D_{i+1}^{(n)^*})x = D_{i+1}^{(n)} D_{i+1}^{(n)^*} x$, we have $x = (D_i^{(n)^*} D_i^{(n)} + D_{i+1}^{(n)} D_{i+1}^{(n)*})^{-1} D_{i+1}^{(n)} D_{i+1}^{(n)^*} x = D_{i+1}^{(n)} D_{i+1}^{(n)^*} (D_i^{(n)^*} D_i^{(n)} + D_{i+1}^{(n)} D_{i+1}^{(n)^*})^{-1} x \in \mathrm{Ran}D_{i+1}^{(n)}$, whence $\mathrm{Ker}D_i^{(n)} = \mathrm{Ran}D_{i+1}^{(n)}$. Therefore, $H_i(T, H) = 0$, $i = 0, 1, \cdots, k$, i.e., $0 \notin \sigma_{\delta,k}(T, H)$. This completes the proof.

Let $A_{ij} \in L(H_1)$, $B \in L(H_2)$, for $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, and $A = \begin{pmatrix} A_{11} \otimes B & \cdots & A_{1n} \otimes B \\ \vdots & & \vdots \\ A_{m1} \otimes B & \cdots & A_{mn} \otimes B \end{pmatrix} \in L(H_1 \otimes H_2 \otimes C^n, H_1 \otimes H_2 \otimes C^m)$, then there exists a unitary operator $U_m \in L(H_1 \otimes H_2 \otimes C^m, H_1 \otimes C^m \otimes H_2)$ to satisfy that $U_m((x_1 \otimes y_1, \cdots, x_m \otimes y_m)) = (x_1, 0, \cdots, 0) \otimes y_1 + \cdots + (0, \cdots, x_m) \otimes y_m$, and such that the following diagram is commuting:

$$\begin{array}{ccc} H_1 \otimes H_2 \otimes C^n & \xrightarrow{A} & H_1 \otimes H_2 \otimes C^m \\ \downarrow U_n & & \downarrow U_m \\ H_1 \otimes C^n \otimes H_2 & \xrightarrow{A' \otimes B} & H_1 \otimes C^m \otimes H_2 \end{array}$$

where $A' = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mn} \end{pmatrix} \in L(H_1 \otimes C^n, H_2 \otimes C^m)$. By the above preparation, we have the following result.

Theorem 2 Suppose H_1, H_2 are the complex Hilbert spaces, $T = (T_1, \cdots, T_n) \in L(H_1)^{(n)}_{\mathrm{com}}$, $T_j' = T_j \otimes I_{H_2} \in L(H_1 \otimes H_2)$, $j = 1, \cdots, n$. $T' = (T'_1, \cdots, T'_n) \in L(H_1 \otimes H_2)^{(n)}_{\mathrm{com}}$, then

(i) $\sigma_{\delta,k}(T', H_1 \otimes H_2) = \sigma_{\delta,k}(T, H_1)$;

(ii) $\sigma_{\pi,k}(T', H_1 \otimes H_2) = \sigma_{\pi,k}(T, H_1)$, where $k \geqslant 0$.

Proof With the above discussion, it is easy to see that the complex $E'(T', H_1 \otimes H_2)$ is isomorphic to the following complex $0 \to H_1 \otimes H_2 \xrightarrow{D_n^{(n)} \otimes I_{H_2}} H_1 \otimes C^n \otimes H_2 \to \cdots \to H_1 \otimes H_2 \to 0$, which has the same exactness with $E'(T, H_1)$: $0 \to H_1 \xrightarrow{D_n^{(n)}} H_1 \otimes C^n \to \cdots \xrightarrow{D_1^{(n)}} H_1 \to 0$, whence (i) and (ii) hold. This completes the proof.

Theorem 3 Let H_j be a complex Hilbert space, $T_j \in L(H_j)$, $j = 1, 2, \cdots, n$, and $H = H_1 \otimes \cdots \otimes H_n$, $\tilde{T}_j = I_{H_1} \otimes \cdots \otimes I_{H_{j-1}} \otimes T_j \otimes I_{H_{j+1}} \otimes \cdots \otimes I_{H_n}$, and $\tilde{T} = (\tilde{T}_1, \cdots, \tilde{T}_n) \in L(H)^{(n)}_{\mathrm{com}}$, then $\prod_{j=1}^{n} \sigma_{\delta,k}(T_j, H_j) \supseteq \sigma_{\delta,k}(\tilde{T}, H) \supseteq \bigcup_{j_1+\cdots+j_n=k} \prod_{i=1}^{n} \sigma_{\delta,j_i}(T_i, H_i)$. For $\sigma_{\pi,k}$, we also have a similar conclusion.

Proof If $n = 1$, then the result is obvious. Suppose that $\sigma_{\delta,k}(\tilde{T}, H) \supseteq \bigcup_{j_1+\cdots+j_{n-1}=k} \prod_{i=1}^{n-1} \sigma_{\delta,j_i}(T_i, H_i)$ holds for $n-1$. Next we will consider the case n, the complex $E'(\tilde{T}, H)$:

$$0 \to \bigotimes_{i=1}^{n-1} H_i \otimes H_n \xrightarrow{\widetilde{D}_n^{(n)}} \bigotimes_{i=1}^{n-1} H_i \otimes H_n \otimes C^n \to \cdots \xrightarrow{\widetilde{D}_1^{(n)}} \bigotimes_{i=1}^{n-1} H_i \otimes H_n \to 0$$

where $\widetilde{D}_k^{(n)} = \begin{pmatrix} \widetilde{D}_k^{(n-1)} & (-1)^{k+1}\mathrm{diag}(\bigotimes_{i=1}^{n-1} I_{H_i} \otimes T_n) \\ 0 & \widetilde{D}_{k-1}^{(n-1)} \end{pmatrix}$

is isomorphic to the complex: $0 \to \bigotimes_{i=1}^{n-1} H_i \otimes H_n \xrightarrow{\widetilde{D}_n^{(n)\prime}} \bigotimes_{i=1}^{n-1} H_i \otimes C^n \otimes H_n \to \cdots \xrightarrow{\widetilde{D}_1^{(n)\prime}} \bigotimes_{i=1}^{n-1} H_i \otimes H_n \to 0$, here

$$\widetilde{D}^{(n)}{}'_k = \begin{pmatrix} D_k^{(n-1)} \otimes I_{H_n} & (-1)^{k+1}\mathrm{diag}(\bigotimes_{i=1}^{n-1} I_{H_i}) \otimes T_n \\ 0 & D_{k-1}^{(n-1)} \otimes I_{H_n} \end{pmatrix}$$

and $D_k^{(n-1)}$ denotes the boundary operators of the complex $E'(T, \bigotimes_{i=1}^{n-1} H_i)$ generated by

$$T = (T_1 \otimes I_{H_2} \otimes \cdots \otimes I_{H_{n-1}}, \cdots, I_{H_1} \otimes \cdots \otimes I_{H_{n-2}} \otimes T_{n-1})$$

Using theorem 1, $0 \notin \sigma_{\delta,k}(\widetilde{T},H) \Leftrightarrow \widetilde{D}_j^{(n)*}\widetilde{D}_j^{(n)} + \widetilde{D}_{j+1}^{(n)}\widetilde{D}_{j+1}^{(n)*}$, $(j=0,1,\cdots,k)$ is invertible $\Leftrightarrow (D_j^{(n-1)*}D_j^{(n-1)} + D_{j+1}^{(n-1)}D_{j+1}^{(n-1)*}) \otimes I_{H_n} + \mathrm{diag}(\bigotimes_{i=1}^{n-1} I_{H_i}) \otimes T_nT_n^*$ and $(D_{j-1}^{(n-1)*}D_{j-1}^{(n-1)} + D_j^{(n-1)}D_j^{(n-1)*}) \otimes I_{H_n} + \mathrm{diag}(\bigotimes_{i=1}^{n-1} I_{H_i}) \otimes T_n^*T_n$, $(j=0,1,\cdots,k)$ are invertible. Therefore, by the hypotheses and theorem 2, it suffices to check the following inclusion: $\sigma_{\delta,k}(\widetilde{T},H) \supseteq \sigma_{\delta,k-1}(T,\bigotimes_{i=1}^{n-1}H_i) \times \sigma_{\delta,1}(T_n,H_n) \cup \sigma_{\delta,k}(T,\bigotimes_{i=1}^{n-1}H_i) \times \sigma_{\delta,0}(T_n,H_n)$.

First, if $T_nT_n^*$ is invertible, then $0 \notin \sigma_{\delta,k}(T,\bigotimes_{i=1}^{n-1}H_i) \times \sigma_{\delta,0}(T_n,H_n)$. If $T_n^*T_n$ is not invertible, then $D_{j-1}^{(n-1)*}D_{j-1}^{(n-1)} + D_j^{(n-1)}D_j^{(n-1)*}$, $(j=1,\cdots,k)$ is invertible, thus $0 \notin \sigma_{\delta,k-1}(T,\bigotimes_{i=1}^{n-1}H_i)$. If $T_n^*T_n$ is invertible, then $0 \notin \sigma_{\delta,1}(T_n,H_n)$. Secondly, if $T_nT_n^*$ is not invertible, then $D_j^{(n-1)*}D_j^{(n-1)} + D_{j+1}^{(n-1)}D_{j+1}^{(n-1)*}$, $(j=0,1,\cdots,k)$ is invertible. Hence, by theorem 1, $0 \notin \sigma_{\delta,k}(T,\bigotimes_{i=1}^{n-1}H_i)$. Therefore, the above inclusion holds.

On the other hand, by theorem 2 and the projection property of Slodkowski joint spectrum, $\sigma_{\delta,k}(\widetilde{T},H) \subseteq \prod_{i=1}^{n}\sigma_{\delta,k}(T_i,H_i)$. This concludes the proof.

The following corollary is a very interesting result about Taylor joint spectrum, left joint spectrum and right joint spectrum.

Corollary 1 Assume that $\widetilde{T},H$ are the above notions in theorem 3, then $\sigma_T(\widetilde{T},H) = \prod_{i=1}^{n}\sigma_T(T_i,H_i) = \prod_{i=1}^{n}\sigma(T_i)$ (See [4]), $\sigma_\pi(\widetilde{T},H) = \prod_{i=1}^{n}\sigma_\pi(T_i)$ and $\sigma_\delta(\widetilde{T},H) = \prod_{i=1}^{n}\sigma_\delta(T_i)$.

Using Slodkowski joint spectral mapping theorem, let $f(z_1,\cdots,z_n) = z_1\cdots z_n$, we get the spectrum, left spectrum and right spectrum of the tensor product operators as follows.

Corollary 2 If $T_j \in L(H_j)$, $(j=1,\cdots,n)$ then $\sigma(\bigotimes_{j=1}^{n}T_j) = \prod_{j=1}^{n}\sigma(T_j)$, $\sigma_\pi(\bigotimes_{j=1}^{n}T_j) = \prod_{j=1}^{n}\sigma_\pi(T_j)$ and $\sigma_\delta(\bigotimes_{j=1}^{n}T_j) = \prod_{j=1}^{n}\sigma_\delta(T_j)$.

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Slodkowski 联合谱和张量积

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摘　要　Slodkowski 联合谱类似于 Taylor 联合谱, 它具有更重要的理论和应用意义. 本文我们刻画了 Slodkowski 联合谱, 同时推广了关于张量积的几个结果.

关键词　算子, 联合谱, 张量积, Koszul 复形

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