

# Common Fixed Point Theorems for Commuting Maps on a Complete Metric Space

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**Abstract:** The concept of  $w$ -distance on a metric space is introduced and three common fixed points theorems for commuting maps on a complete metric space are proved. These results extended fixed point theorems of Jungck and Ćirić.

**Key words:** distance, fixed point, complete metric space

## 1 Preliminaries

The purpose of this paper is to generalize and unify fixed point theorems of Jungck<sup>[1]</sup> and Ćirić<sup>[2]</sup>. Throughout the paper we denote by  $N$  the set of all positive integers. Maps  $f, g: X \rightarrow X$  are said to commute iff  $fg = gf$ .

**Definition 1** Let  $X$  be a metric space with metric  $d$ . Then a function  $p: X \times X \rightarrow [0, \infty)$  is called a  $w$ -distance on  $X$  if the following are satisfied:

- 1) For any  $x, y, z \in X$ , there exists  $p(x, z) \leq p(x, y) + p(y, z)$ ;
- 2) For any  $x \in X, p(x, \cdot): X \rightarrow [0, \infty)$  is lower semicontinuous;
- 3) For any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $p(z, x) \leq \delta$  and  $p(z, y) \leq \delta$  imply  $d(x, y) \leq \varepsilon$ .

Let  $X$  be a complete metric space,  $f$  be a continuous self-map on  $X$  and  $g$  be any self-map on  $X$  such that  $g(X) \subset f(X)$ . We define a sequence of points  $\{x_n\}$  as follows. For  $x_0$  is an arbitrary point in  $X$ . Let  $x_1 \in X$ , guaranteed by  $g(X) \subset f(X)$ , be such that  $g(x_0) = f(x_1)$ . Having defined  $x_n \in X$ , let  $x_{n+1} \in X$  be such that  $g(x_n) = f(x_{n+1})$ .

Let  $g(x_n)(= f(x_{n+1})) = y_n (n = 0, 1, 2, \dots)$  we denote by  $O(y_k; n)$  the set of points  $\{y_k, y_{k+1}, \dots, y_{k+n}\}$ . Assume that  $f$  and  $g$  satisfy the following inequality. There exists a constant  $q \in (0, 1)$  such that for every  $x, y$  in  $X$

$$p(gx, gy) \leq q \max \{p(fx, fy), p(fx, gx), p(fy, gy), p(fx, gy), p(fy, gx)\} \quad (1)$$

For  $A \subset X$ , let  $\delta(A) = \sup\{p(x, y): x, y \in A\}$ . The following lemmas are fundamental.

**Lemma 1**<sup>[3]</sup> Let  $X$  be metric space with metric

$d$  and let  $p$  be a  $w$ -distance on  $X$ . Let  $\{x_n\}$  and  $\{y_n\}$  be sequences in  $X$ , let  $\{\alpha_n\}$  and  $\{\beta_n\}$  be sequences in  $[0, \infty)$  converging to 0, and let  $x, y, z \in X$ . Then the following hold.

- (i) If  $p(x_n, y) \leq \alpha_n$  and  $p(x_n, z) \leq \beta_n$  for any  $n \in N$ , then  $y = z$ . In particular, if  $p(x, y) = 0$  and  $p(x, z) = 0$ , then  $y = z$ ;
- (ii) If  $p(x_n, y_n) \leq \alpha_n$  and  $p(x_n, z) \leq \beta_n$  for any  $n \in N$ , then  $\{y_n\}$  converges to  $z$ ;
- (iii) If  $p(x_n, x_m) \leq \alpha_n$  for any  $n, m \in N$  with  $m > n$ , then  $\{x_n\}$  is a cauchy sequence;
- (iv) If  $p(y, x_n) \leq \alpha_n$  for any  $n \in N$ , then  $\{x_n\}$  is a cauchy sequence.

**Lemma 2** For  $k \geq 0$  and  $n \in N$ , suppose  $\delta(O(y_k; n)) > 0$ , then  $\delta(O(y_k; n)) = p(y_k, y_j)$ , where  $j$  is such that  $k < j \leq k + n$ . Also

$$\delta(O(y_k; n)) \leq q\delta(O(y_{k-1}; n+1)) \quad k \geq 1 \quad (2)$$

**Proof** For  $i, j$  such that  $y_i, y_j \in X (1 \leq i < j)$ , applying inequality (1), we get

$$\begin{aligned} p(y_i, y_j) &= p(gx_i, gx_j) \\ &\leq q \max \{p(fx_i, fx_j), p(fx_i, gx_i), p(fx_j, gx_j), p(fx_i, gx_j), p(fx_j, gx_i)\} \\ &= q \max \{p(y_{i-1}, y_{j-1}), p(y_{i-1}, y_i), p(y_{j-1}, y_j), p(y_{i-1}, y_j), p(y_{j-1}, y_i)\} \end{aligned}$$

Thus

$$p(y_i, y_j) \leq q\delta(O(y_{i-1}; j-i+1)) \quad (3)$$

Let  $\delta(O(y_k; n)) = p(y_i, y_j)$ , for some  $i, j$  satisfying  $k \leq i < j \leq k + n$ .

If  $i > k$ , then by (3)

$$\delta(O(y_k; n)) \leq q\delta(O(y_{i-1}; j-i+1))$$

with  $i-1 \geq k$  and  $j \leq k + n$ , whence  $\delta(O(y_k; n)) \leq q\delta(O(y_k; n))$ , a contradiction. This proves the

first assertion. Moreover

$$\begin{aligned}\delta(O(y_k, n)) &= p(y_k, y_j) \\ &\leq q\delta(O(y_{k-1}; j - k + 1)) \\ &\leq q\delta(O(y_{k-1}; n + 1))\end{aligned}$$

which proves lemma 2.

**Lemma 3** Under the hypotheses of lemma 2

$$\delta(O(y_k; n)) \leq \frac{q^K}{1-q} p(y_0, y_1) \quad (4)$$

$$\begin{aligned}\textbf{Proof} \quad \delta(O(y_e; m)) &= p(y_e, y_j) \\ &\leq p(y_e, y_{e+1}) + p(y_{e+1}, y_j) \\ &\leq p(y_e, y_{e+1}) + \delta(O(y_{e+1}; m - 1))\end{aligned}$$

since  $j \leq e + m$ . Thus  $\delta(O(y_e; m)) \leq p(y_e, y_{e+1}) + q\delta(O(y_e; m))$  in view of (2). This leads to

$$\delta(O(y_e; m)) \leq \frac{1}{1-q} p(y_e, y_{e+1}) \quad (5)$$

By repeated application of (2) we have

$$\delta(O(y_k; n)) \leq q^k \delta(O(y_0, n + k))$$

whence (4) follows in view of (5) with  $e = 0$ ,  $m = n + k$ , thus

$$\begin{aligned}\delta(O(y_k; n)) &\leq q^k \delta(O(y_0, n + k)) \\ &\leq q^k \frac{p(y_0, y_1)}{1-q}\end{aligned}$$

## 2 Main Results

**Theorem 1** Let  $X$  be complete metric space,  $f$  be a continuous self-map on  $X$  and  $g$  be any self-map on  $X$  that commutes with  $f$ . Further let  $f$  and  $g$  satisfy  $g(X) \subset f(X)$  and (1). Then  $f$  and  $g$  have a unique common fixed point.

**Proof** We first proof that it is sufficient to produce a point  $y$  such that  $f(y) = g(y)$ . If for some  $n$  and  $k$ ,  $\delta(O(y_k; n)) = 0$ , we have  $y_k = y_{k+1}$ , i.e.,  $f(x_{k+1}) = g(x_{k+1})$ . Otherwise  $\delta(O(y_k; n)) > 0$ . For  $m, n, k \in N$  and  $m > n > k$

$$\begin{aligned}p(y_n, y_m) &\leq \delta(O(y_k; m - k)) \\ &\leq \frac{q^k}{1-q} p(y_0, y_1) \rightarrow 0 \quad k \rightarrow \infty\end{aligned}$$

in view of lemma 3.

From lemma 1,  $\{y_n\}$  is a cauchy sequence, has a limit, say  $y$ . By continuity of  $f$ ,  $\{f(y_n)\}$  converges to  $f(y)$ . Moreover,  $\{g(y_n) = f(y_{n+1})\}$  also converges to  $f(y)$ .

Applying inequality (1) we get

$$\begin{aligned}p(fy_{n+1}, gy) &= p(gy_n, gy) \leq q\max\{p(fy_n, fy), \\ p(fy_n, gy_n), p(fy, gy), p(fy_n, gy), p(fy_n, gy_n)\}\end{aligned}$$

in the limit leads to

$$p(fy, gy) \leq q\max\{p(fy, gy), p(fy, fy)\} \quad (6)$$

Also applying inequality (1) we get

$$p(fy_{n+1}, fy_{n+1}) = p(gy_n, gy_n)$$

$$\begin{aligned}&\leq q\max\{p(fy_n, fy_n), p(fy_n, gy_n), p(fy_n, gy_n), \\ p(fy_n, gy_n), p(fy_n, gy_n)\} \\ &= q\max\{p(fy_n, fy_n), p(fy_n, gy_n)\}\end{aligned}$$

in the limit leads to

$$p(fy, fy) \leq q\max\{p(fy, fy), p(fy, gy)\} \quad (7)$$

From (6), (7) we get  $p(fy, fy) = 0$ ,  $p(fy, gy) = 0$ .

By lemma 1, we have  $fy = gy$ .

Applying inequality (1) and  $fy = gy$ ,  $fgy = gfy$ , we get  $p(ggy, ggy) \leq q\max\{p(fgy, fgy), p(fgy, ggy), p(fgy, ggy), p(fgy, ggy), p(fgy, ggy)\} = qp(ggy, ggy)$ , and hence  $p(ggy, ggy) = 0$ .

Again applying inequality (1) and  $fy = gy$ ,  $p(ggy, ggy) = 0$ , we get

$$\begin{aligned}p(ggy, gy) &\leq q\max\{p(gy, gy), p(gy, ggy)\} \\ p(gy, ggy) &\leq q\max\{p(gy, ggy), p(ggy, gy)\}\end{aligned}$$

from which it follows that

$$p(ggy, gy) = 0, p(gy, ggy) = 0$$

By lemma 1, we have  $ggy = gy$ , showing that  $g(y)$  is a fixed point of  $g$ .

Observing  $f(gy) = g(fy) = g(gy) = gy$ , we see that  $g(y)$  is also a fixed point of  $f$ . To prove uniqueness, let  $u = fu$ ,  $u = gu$  and  $v = fv$ ,  $v = gv$ . Then, by (1), we have

$$\begin{aligned}p(u, u) &= p(gu, gu) \leq q\max\{p(fu, fu), \\ p(fu, gu), p(fu, gu), p(fu, gu), \\ p(fu, gu)\} &= qp(u, u).\end{aligned}$$

Since  $0 < q < 1$ , it follows that  $p(u, u) = 0$ , similarly we obtain  $p(v, v) = 0$ .

Applying inequality (1) and  $p(u, u) = 0$ ,  $p(v, v) = 0$ , we get

$$\begin{aligned}p(u, v) &= p(gu, gv) \\ &\leq q\max\{p(fu, fv), p(fu, gu), \\ p(fv, gv), p(fu, gv), p(fv, gu)\} \\ &= q\max\{p(u, v), p(v, u)\} \\ p(v, u) &= p(gv, gu) \\ &\leq q\max\{p(fv, fu), p(fv, gv), \\ p(fu, gu), p(fv, gu), p(fu, gv)\} \\ &= q\max\{p(v, u), p(u, v)\}\end{aligned}$$

Since  $0 < q < 1$ , it follows that  $p(u, v) = 0, p(v, u) = 0$ .

By lemma 1, we have  $u = v$ .

**Theorem 2** Let  $X$  be a complete metric space,  $f$  be a self-map on  $X$  such that  $f^2$  is continuous. Let  $g: f(X) \rightarrow X$  be such that

$$gf(X) \subset f^2(X) \quad (8)$$

and  $f(g(x)) = g(f(x))$  whenever both sides are defined. Further, suppose there exists a number  $q \in (0, 1)$  such that (1) holds for every  $x, y$  in  $f(X)$ . Then  $f$  and  $g$  have a unique common fixed point.

**Proof** Starting with an arbitrary point  $x_0$  in  $f(X)$  and appealing to condition (8), we construct a

sequence  $\{x_n\}$  of points in  $f(X)$  such that  $f(x_{n+1}) = g(x_n) = y_n$ . Note that  $f(y_n) = f(g(x_n)) = g(f(x_n)) = g(y_{n-1}) = z_n$ . Arguing as lemmas 2 and 3, we get that for  $k \geq 0, n \in N$ ,

$$\delta(O(z_k, n)) \leq \frac{q^k}{1-q} p(z_0, z_1)$$

As the proof of theorem 1,  $\{z_n\}$  is a Cauchy sequence in  $X$  and hence convergent to some  $z$  in  $X$ . By continuity of  $f^2$ ,  $\{f^2(z_n)\}$  converges to  $f^2(z)$ .

Moreover,  $gf(z_n) = gf(f^2 x_{n+1}) = f^2(f(gx_{n+1})) = f^2(z_{n+1})$  implies that  $\{gf(z_n)\}$  converges to  $f^2(z)$ . By (1) we have

$$\begin{aligned} p(f^2 z_{n+1}, f^2 z_{n+1}) &= p(gfz_n, gfz_n) \\ &\leq q \max\{p(f^2 z_n, f^2 z_n), p(f^2 z_n, gfz_n)\} \\ p(f^2 z_{n+1}, gfz) &= p(gfz_n, gfz) \\ &\leq q \max\{p(f^2 z_n, f^2 z), p(f^2 z_n, gfz_n), \\ &\quad p(f^2 z, gfz), p(f^2 z_n, gfz), p(f^2 z, gfz_n)\} \end{aligned}$$

Let  $n$  tend to infinity, we have

$$\begin{aligned} p(f^2 z, f^2 z) &\leq q \max\{p(f^2 z, f^2 z), p(f^2 z, gfz), \\ &\quad p(f^2 z, gfz)\} \\ &\leq q \max\{p(f^2 z, f^2 z), p(f^2 z, gfz)\} \end{aligned}$$

from which it follows that

$$p(f^2 z, f^2 z) = 0, p(f^2 z, gfz) = 0$$

By lemma 1, we have  $f^2 z = gfz$ . By (1) we also have

$$\begin{aligned} p(g(gfz), gfz) &\leq q \max\{p(f(gfz), f^2 z), \\ &\quad p(f(gfz), g(fz)), p(f^2 z, g(fz)), \\ &\quad p(f(gfz), g(fz)), p(f^2 z, g(gfz))\} \\ &= q \max\{p(g(gfz), gfz), p(gfz, g(gfz))\} \\ p(g(fz), g(gfz)) &\leq q \max\{p(f^2 z, f(gfz)), \\ &\quad p(f^2 z, gfz), p(f(gfz), g(gfz)), \end{aligned}$$

$$\begin{aligned} &p(f^2 z, g(gfz)), p(f(gfz), gfz)\} \\ &= q \max\{p(gfz, g(gfz)), p(g(gfz), gfz)\} \end{aligned}$$

Since  $0 < q < 1$ , it follows that  $p(gfz, g(gfz)) = 0, p(g(gfz), gfz) = 0$ .

Since  $p(gfz, gfz) = 0$ , by lemma 1, we have  $g(gfz) = gfz$ . Show that  $gfz$  is a fixed point of  $g$ . Observing  $g(gfz) = g(f^2 z) = f(gfz) = gfz$ , we see that  $gfz$  is also a fixed point of  $f$ . The uniqueness follows once again from theorem 1.

As an easy corollary of this we have the following theorem.

**Theorem 3** Let  $X$  be a complete metric space.

Let  $f$  and  $g$  be commuting self-maps on  $X$  such that  $g(X) \subset f(X)$ . Further, let  $f^2$  be continuous and there exists a constant  $q \in (0, 1)$  such that for every  $x, y$  in  $X$

$$p(gx, gy) \leq qp(fx, fy)$$

Then  $f$  and  $g$  have a unique common fixed point.

**Proof** Note  $f(X) \supset g(X)$  implies that  $f^2(X) \supset f(g(X)) = g(f(X))$ , i.e., (8) holds. Since (1) obviously holds, the conclusion follows in view of theorem 2.

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# 完备度量空间上交换映射的公共不动点定理

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**摘要** 在度量空间中引入  $w$ -距离, 证明了在完备度量空间中交换映射的 3 个公共不动点定理. 这些结果推广了 Jungck and Ćirić 的不动点定理.

**关键词** 距离, 不动点, 完备度量空间

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