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Circular Chromatic Numbers of Some Distance Graphs

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Abstract: The circular chromatic number of a graph is an important parameter of a graph. The distance graph $G(Z, D)$, with a distance set D, is the infinite graph with vertex set $Z = \{0, \pm 1, \pm 2, \cdots\}$ in which two vertices x and y are adjacent iff $|y - x| \in D$. This paper determines the circular chromatic numbers of two classes of distance graphs $G(Z, D_{m,k,k+1})$ and $G(Z, D_{m,k,k+1,k+2})$.

Key words: distance graph, fractional chromatic number, circular chromatic number

The circular chromatic number and the fractional chromatic number of a graph are two natural generalizations of the chromatic number of a graph.

Definition 1 Suppose p and q are positive integers such that $p \geqslant 2q$. A (p, q)-coloring of a graph $G = (V, E)$ is a mapping c from V to $\{0, 1, \cdots, p-1\}$ such that $\| c(y) - c(x) \|_p \geqslant q$ for any edge xy in E, where $\| a \|_p = \min\{a, p - a\}$. The circular chromatic number $\chi_c(G)$ of G is the infimum of the ratios p/q for which there exist (p, q)-colorings of G.

Definition 2 Suppose Γ is the set of all independent sets of a graph $G = (V, E)$. A fractional coloring of a graph G is a mapping c from Γ to $[0, 1]$ such that $\sum_{S \in \Gamma, x \in S} c(S) = 1$ for every vertex x of G. The fractional chromatic number $\chi_f(G)$ of G is the infimum of the values $\sum_{S \in \Gamma} c(S)$ of fractional colorings c of G.

For equivalent definitions of the circular chromatic number and the fractional chromatic number of a graph, see Refs. [1,2].

For any graph G, let $\omega(G), \alpha(G), |G|$ denote the clique number, the independence number, and the vertex number of G. From the Refs. [3,4], we know that

$$\max\left\{\omega(G), \frac{|G|}{\alpha(G)}\right\} \leqslant \chi_f(G) \leqslant \chi_c(G) \leqslant \chi(G) = \lceil \chi_c(G) ? \quad (1)$$

Definition 3 Let D be a finite integer set. A distance graph $G(Z, D)$ is such an infinite graph with vertex set $Z = \{0, \pm 1, \pm 2, \cdots\}$ and edge set $E = \{xy \| x - y | \in D, x \in Z, y \in Z\}$.

Now we assume $k + 2 \leqslant 2k - 1$. Some results concerning the chromatic number of distance graphs were presented in Refs. [3 - 7]. Let distance set $D_{m,k,k+1,k+2} = \{1, 2, \cdots, m\} - \{k, k+1, k+2\}$, $D_{m,k,k+1} = \{1, 2, \cdots, m\} - \{k, k+1\}$ and $D_{m,k} = \{1, 2, \cdots, m\} - \{k\}$. The chromatic number, the fractional chromatic number and the circular chromatic number of graph $G(Z, D_{m,k})$ were determined by G. J. Chang[3,4]. The chromatic numbers and the fractional chromatic numbers of graph $G(Z, D_{m,k,k+1})$ and graph $G(Z, D_{m,k,k+1,k+2})$ were determined in Ref. [8]. This paper discusses the circular chromatic numbers of graph $G(Z, D_{m,k,k+1})$ and $G(Z, D_{m,k,k+1,k+2})$.

1 The Circular Chromatic Numbers of $G(Z, D_{m,k,k+1})$ and $G(Z, D_{m,k,k+1,k+2})$

The circular chromatic number of the distance graph $G(Z, D_{m,k,k+1,k+2})$ is discussed in this section. For simplicity, let χ_f, χ_c, χ respectively denote the fractional chromatic number, the circular chromatic number and the chromatic number of the distance graph $G(Z, D_{m,k,k+1,k+2})$. Ref. [8] determined the circular chromatic numbers of distance graph $G(Z, D_{m,k,k+1,k+2})$ for some pairs of integers m and k. theorems 1 - 3 present the results.

Theorem 1 If $m < 2k$, then $\chi_f = \chi_c = \chi = k$.

Theorem 2

$$\chi_f = \chi_c = \chi = \begin{cases} k+1 & m = 2k+1 \\ k+2 & m = 2k+3 \end{cases}$$

Theorem 3 Suppose $m \geqslant 2k + 4$, $m + k + 1 = 2^r m'$ and $k = 2^s k'$, where r and s are non-negative integers and m' and k' are odd integers. If $r > s$, then $\chi_f = \chi_c = \chi = (m + k + 1)/2$.

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We shall determine the circular chromatic numbers of the distance graph $G(Z, D_{m,k,k+1,k+2})$ for other pairs of integers m and k.

Theorem 4 If $m = 2k$ or $2k+2$, then $\chi_f = \chi_c = (m+1)/2$.

Proof By theorem 3[8], $\chi_f = (m+1)/2$ when $m = 2k$ or $2k+2$. To prove theorem 4, it is sufficient to prove that $\chi_c \leqslant (m+1)/2$. Now $m/2$ is relatively prime to $m+1$ and $m(m-1)/2 \equiv 1 \pmod{m+1}$. Consider the mapping c defined by $c(i) = (m-1)i \pmod{m+1}$ for all $i \in Z$. For any edge ij in $G(Z, D_{m,k,k+1,k+2})$, we shall prove that $\| c(i) - c(j) \|_{m+1} \geqslant 2$. Suppose to the contrary that $c(i) - c(j) \equiv 0, 1, -1 \pmod{m+1}$. Then we have $i - j \equiv 0, k, -k \pmod{m+1}$ when $m = 2k$, and $i - j \equiv 0, k+1, -k-1 \pmod{m+1}$ when $m = 2k+2$, it follows that in either case we have $ij \notin E(G)$, which contradicts the fact that i is adjacent to j. Thus c is an $(m+1, 2)$-coloring of $G(Z, D_{m,k,k+1,k+2})$, whence $\chi_c \leqslant (m+1)/2$.

In the following we denote the subgraph of $G(Z, D_{m,k,k+1,k+2})$ induced by $V_i = \{0, 1, \cdots, i\}$ as G_i.

Lemma 1[4] If G has a circular chromatic number p/q (where p and q are relatively prime), then $p \leqslant |V(G)|$, and any (p, q)-coloring c of G is an onto mapping from $V(G)$ to $\{0, 1, \cdots, p-1\}$.

Theorem 5 If $m \geqslant 2k+4$ and $(m+k+1, k) = 1$, then $\chi_f = \chi_c = (m+k+1)/2$.

Proof Consider the graph G_{m+k}, $\alpha(G_{m+k}) = 2$ when $m \geqslant 2k+4$. According to (1), we have $\chi_f(G_{m+k}) \geqslant (m+k+1)/2$. Since graph G_{m+k} is a subgraph of the distance graph $G(Z, D_{m,k,k+1,k+2})$, it follows that $\chi_f \geqslant \chi_f(G_{m+k}) = (m+k+1)/2$.

On the other hand, by theorem 10[3], we know that if $m \geqslant 2k+4$ and $(m+k+1, k) = 1$ then $\chi_f(G(Z, D_{m,k})) = \chi_c(G(Z, D_{m,k})) = (m+k+1)/2$ when $(m+k+1, k) = 1$. Since graph $G(Z, D_{m,k,k+1,k+2})$ is a subgraph of $G(Z, D_{m,k})$, so $\chi_c \leqslant \chi_c(G(Z, D_{m,k})) = (m+k+1)/2$.

According to (1), $\chi_f = \chi_c = (m+k+1)/2$.

Lemma 2 Suppose $m \geqslant 2k+4$, $m+k+1 = 2^r m'$ and $k = 2^s k'$, where r and s are integers, m' and k' are odd integers. If $1 \leqslant r \leqslant s$, then $\chi_c(G_{m+2k-1}) > (m+k+1)/2$.

Proof Since $m+k+1$ is even and $\chi_c(G_{m+2k-1}) > \chi(G_{m+2k-1}) - 1$, it suffices to show that $\chi(G_{m+2k-1}) > (m+k+1)/2$. Assume to the contrary that $\chi(G_{m+2k-1}) \leqslant (m+k+1)/2$, and that c is an $(m+k+1)/2$-coloring of G_{m+2k-1}.

For each integer i with $0 \leqslant i \leqslant k-2$, consider the subgraph of G_{m+2k-1} induced by the $m+k+1$ vertices $\{i, i+1, \cdots, i+m+k\}$. Then the subgraph has an independence number 2. Therefore, each of the $(m+k+1)/2$ colors is used at most, and thus exactly twice in this subgraph. Consequently, vertices i and $i+m+k+1$ have the same colors for any i. So vertices $i+k+1$ and $i+k+2$ are adjacent to vertex $i+m+k+1$. Therefore, for each $j \in S = \{0, 1, \cdots, m+k\}$, the only possible vertices in S having the same color as j are $j+k$ and $j-k$.

Consider the circulant graph $C(m+k+1, k)$, with vertex set S and in which vertex i is adjacent to vertex j iff $j \equiv i+k$ or $i-k \pmod{m+k+1}$. It follows from the discussion in the preceding paragraph that the two vertices x and y in S have the same color only if xy is an edge of graph $C(m+k+1, k)$. Since the intersection of each color class with S contains exactly two vertices, the coloring induces a perfect matching of $C(m+k+1, k)$. However, $C(m+k+1, k)$ is the disjoint union of d cycles of length $(m+k+1)/d$, where $d = \gcd(m+k+1, k)$. Since $C(m+k+1, k)$ has a perfect matching, each cycle has an even length. This implies that $r > s$, contrary to the assumption $r \leqslant s$. Hence, $\chi_c(G_{m+2k-1}) > (m+k+1)/2$.

Lemma 3 Suppose $m \geqslant 2k+4$. If $m+k+1$ is odd and $\gcd(m+k+1, k) \neq 1$, then $\chi_c(G_{m+2k-1}) > (m+k+1)/2$.

Proof Let H_i $(0 \leqslant i \leqslant k-2)$ denote the subgraph of G_{m+2k-1} induced by consecutive $m+k+1$ vertices $\{i, i+1, \cdots, i+m+k\}$ and clearly $\chi_c(H_i) \geqslant (m+k+1)/\alpha(H_i) = (m+k+1)/2$.

Suppose $\chi_c(G_{m+2k-1}) = (m+k+1)/2$. Since $m+k+1$ and 2 are relatively prime, every $(m+k+1, 2)$-coloring c of G_{m+2k-1}, which is also an $(m+k+1, 2)$-coloring of H_i, is onto and hence one-to-one on graph H_i. Thus, there exists an ordering $x_0, x_1, \cdots, x_{m+k}, x_0$ of $V(H_i)$ such that $c(x_j) = j$ for $0 \leqslant j \leqslant m+k$. So $X = (x_0, x_1, \cdots, x_{m+k}, x_0)$ is a cycle in the complement H'_i of H_i. At the same time, due to vertices i and $i+m+k+1$ have the same colors, the only possible vertices having the color $c(i) \pm 1$ must be $i+k$ and $i+m+1 \equiv i-k \pmod{m+k+1}$. It means each $x_i x_{i+1} \in X$ satisfies $x_i - x_{i+1} \equiv \pm k \pmod{m+k+1}$.

For graph H_0, let $m = ak+b$, where $0 \leqslant b < k$. The following paths must be on the cycle X:

$P_i: i, k+i, 2k+i, \cdots, ak+i, (a+1)k+i$ $\quad 0 \leqslant i \leqslant b$

$P_j: j, k+j, 2k+j, \cdots, ak+j$

$b+1 \leqslant j \leqslant k-1$

It is easy to see that $P_b P_{k-1}$ is a path of the cycle X, so is $P_{b-1}P_{k-2}$. Continuing this process, we have that $P'_t = P_{b+1+t}P_t$, where the index $b+1+t$ is taken modulo k, is a path of the cycle X for $0 \leqslant t \leqslant k-1$. Since $\gcd(m+k+1, k) \neq 1$, we have $\gcd(b+1, k) \neq 1$. Therefore, these paths P'_t form at least two disjoint cycles, contrary to the assumption that X is a cycle. Thus, the coloring c does not exist and $\chi_c(G_{m+2k-1}) > (m+k+1)/2$.

Theorem 6 Suppose $m \geqslant 2k+4$, $m+k+1 = 2^r m'$ and $k = 2^s k'$, where r and s are integers and m' and k' are odd integers. If $r \leqslant s$ and $\gcd(m+k+1, k) \neq 1$, then $\chi_c = (m+k+2)/2$.

Proof Since theorem 3.6[3] showed that $\chi_c(G(Z, D_{m,k})) \leqslant (m+k+2)/2$ when $m \geqslant 2k$, and the distance graph $G(Z, D_{m,k,k+1,k+2})$ is a subgraph of the distance graph $G(Z, D_{m,k})$, then we have $\chi_c \leqslant \chi_c(G(Z, D_{m,k})) \leqslant (m+k+2)/2$.

On the other hand, suppose $\chi_c(G_{m+2k-1}) = p/q$, where p and q are relatively prime. Then, $p \leqslant |G_{m+2k-1}| = m+2k$ and $p/q > (m+k+1)/2$ according to lemma 2 and lemma 3. If $q \geqslant 3$, then $p > (m+k+1)q/2 \geqslant 3(m+k+1)/2 > m+2k$, a contradiction. Hence, $q \leqslant 2$ and so $\chi_c \geqslant \chi_c(G_{m+2k-1}) = p/q \geqslant (m+k+2)/2$. Thus, $\chi_c = (m+k+2)/2$.

Similarly, we can prove theorems 7 – 9.

Theorem 7 If $m < 2k$, then $\chi_c(G(Z, D_{m,k,k+1})) = k$.

Theorem 8 If $m = 2k, 2k+1$, then $\chi_c(G(Z, D_{m,k,k+1})) = (m+1)/2$.

Theorem 9 When $m \geqslant 2k+2$, $m+k+1 = 2^r m'$ and $k = 2^s k'$, where r and s are integers, m' and k' are odd integers. If $r \leqslant s$ and $\gcd(m+k+1, k) \neq 1$, then $\chi_c(G(Z, D_{m,k,k+1})) = (m+k+2)/2$; otherwise, $\chi_c(G(Z, D_{m,k,k+1})) = (m+k+1)/2$.

Remarks By using similar method, we may determine the fractional chromatic number, the circular chromatic number and the chromatic number of the distance graph $G(Z, D)$ with the distance set $D = \{1, 2, \cdots, m\} - \{k, k+1, \cdots, k'\}$, where $k' \leqslant 2k-1$.

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一类距离图的圆色数

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摘　要　圆色数是图的一个重要参数.距离图 $G(Z,D)$是具有顶点集 $Z=\{0,\pm1,\pm2,\cdots\}$、距离集 D,且满足顶点 x 与 y 相邻的充要条件是$|y-x|\in D$ 的无限图.本文确定了两类距离图 $G(Z,D_{m,k,k+1})$和 $G(Z,D_{m,k,k+1.k+2})$的圆色数.

关键词　距离图,分式色数,圆色数

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