

Hamilton Graphs Involving Connectivity

Zhou Xiaoyue^{1*} Huang Yuenian²

(¹Nanjing Architectural and Civil Engineering Institute, Nanjing 210009, China)

(²Jinling Petrochemical Design Institute, Nanjing 210042, China)

Abstract: Let G be a 2-connected simple graph of order n and connectivity k . Bauer, Broersma and Li proved that for an independent set $S = \{u, v, w\}$, $d(u) + d(v) + d(w) \geq n + k$, then G is Hamiltonian. This paper improves the result. Let S be an independent set. If there exist $u, v \in S$, $d(u, v) = 2$, then S is called a 2-independent set. This paper proves the following result. Let G be a simple graph of order n and connectivity $k \geq 2$. If for every 2-independent set $S = \{u, v, w\}$, $d(u) + d(v) + d(w) \geq n + k$, then G is Hamiltonian. This result implies that we may consider all triples of 2-independent set instead of all triples of independent set.

Key words: connectivity, independent set, Hamilton graph

This paper uses terms and notation of Ref. [1]. Throughout, G denotes an undirected connected simple graph of order n (≥ 3) with connectivity k and independence number α . Let L be a subset of $V(G)$, F a subgraph of G and v a vertex in G . Define

$$N_L(v) = \{u \mid u \in L, uv \in E(G)\}$$

$$N_L(F) = \bigcup_{v \in V(F)} N_L(v)$$

$$N^k(v) = \{u \mid u \in V \text{ and } d(u, v) = k\}$$

For the special case when $L = V(G)$, we simply write $N(v)$, $N(F)$. If no ambiguity, we sometimes write F instead of $V(F)$. Define $\Delta(L) = \max\{d(u) \mid u \in L\}$ and $\sum(L) = \sum_{u \in L} d(u)$. Let S be an

independent set of G . Put $r = \min\{d(u, v) \mid u, v \in S\}$. S is called an r -independent set.

If P is a path of a graph G , we consider P with a given orientation. Let $x, y \in V(P)$. By xPy we denote the consecutive vertices on P from x to y in the given orientation. The same vertices, in reverse order are given by yPx . We consider xPy and yPx both as paths and as vertex sets. Analogous notation is used with respect to cycles instead of paths. We use x^+ to denote the successor vertex of x on P and x^- to denote its predecessor vertex. For any $B \subseteq V(P)$, defines $B^+ = \{v^+ \mid v \in B\}$.

It is well known that there are many sufficient conditions of Hamilton graphs. Degree condition has long been fundamental tools in the study of Hamilton graphs. Some papers discuss degree sum of triples of independent vertices. One of these results is as follows.

Theorem 1^[2] Let G be a simple graph of order n and connectivity $k \geq 2$ such that $d(u) + d(v) + d(w) \geq n + k$, for any independent set $\{u, v, w\}$,

then G is Hamiltonian.

Recently, Wei^[3] gave a new proof of theorem 1. Inspired by Fan's type condition in Ref. [4], we consider all triples of 2-independent set instead of all triples of independent set. The following result is a generalization of theorem 1.

Theorem 2 Let G be a simple graph of order n and connectivity $k \geq 2$. If for every 2-independent set $S = \{u, v, w\}$, $d(u) + d(v) + d(w) \geq n + k$, then G is Hamiltonian.

In order to prove theorem 2, we first give a generalization of Bondy's result.

Theorem 3 Let G be a simple graph of order n and connectivity $k \geq 2$. If for every 2-independent set $S = \{u, v, w\}$, $d(u) + d(v) + d(w) \geq n + 2$, then every longest cycle in G is a dominating cycle.

Now we give some properties for the longest cycle in a graph G .

Let C be a longest cycle in G with a fixed direction and $|C| < n$. Let H be any component of $G \setminus V(C)$ and $N_C(H) = \{v_1, v_2, \dots, v_m\}$, where v_i occurs on C in the order of their indices. For any j ($1 \leq j \leq m$), let x_j be any vertex in H which is adjacent to v_j . It is possible that $x_i = x_j$ for some $i \neq j$. Since G is k -connected, we have that $m \geq k \geq 2$. Set $u_i = v_i^+$, $w_i = v_{i+1}^-$ and $C_i = u_i C w_i$ for $1 \leq i \leq m$. Denote by $A_i(B_i)$ the set of vertices $a_i(b_i)$ such that there exists an $a_i w_i$ -path ($u_i b_i$ -path) in G with vertex set C_i for $1 \leq i \leq m$. Obviously, $y^- \in A_i$ if $y \in N(u_i) \cap C_i$ and $y^+ \in B_i$ if $y \in N(w_i) \cap C_i$. Put $A = \{a_1, a_2, \dots, a_m\}$ and $B = \{b_1, b_2, \dots, b_m\}$, where a_i is any vertex in A_i and b_i is any vertex in B_i for $1 \leq$

$i \leq m$. Since C is a longest cycle, we have the following lemmas.

Lemma 1 For any $x \in V(H)$, $A \cup \{x\}$ and $B \cup \{x\}$ are independent, and for any $u, v \in A \cup \{x\}$ or $B \cup \{x\}$, $N(u) \cap N(v) \subseteq V(C)$.

Lemma 2 Let $a_i \in A_i$ and $a_j \in A_j (i < j)$. Then for any $v \in N(a_i) \cap V(v_{i+1}Cv_j)$, $a_j v^- \notin E$; and for any $v \in N(a_i) \cap V(v_{j+1}Cv_j)$, $a_j v^+ \notin E$. Symmetrically, we have similar property for $b_i \in B_i$ and $b_j \in B_j (i \neq j)$.

Lemma 3 Let $a_i \in A_i$ and $b_j \in B_j (i < j)$ and $a_i \neq a_j$. Then $v^+ b_j \notin E$ and $v^- b_j \notin E$ for any $v \in N(a_i) \cap V(v_{j+1}Cv_j)$.

Proof of Theorem 3 The proof is by contradiction. Suppose C is not a dominating cycle of G and H is a nontrivial component in $G - V(C)$. It follows from G is 2-connected that there exist two integers r, s , $1 \leq r \leq s \leq m$, such that $x_r \neq x_s$. Since C is a longest cycle, Lemmas 1 – 3 hold. Moreover, we have the following claims.

Claim 1 If $v \in u_r C u_s$ and $u_r v \in E$, then $u_s v^{++} \notin E$. Similarly, if $v \in u_s C u_r$ and $u_r v \in E$, then $u_s v^{++} \notin E$. Specially, $u_r u^+ \notin E$ and $u_s^+ u \notin E$.

Let $S = \{x_r, u_r, u_s\}$, then S is a 2-independent set. Define

$$R_1 = \{v \in u_r C u_s \mid u_r v^+ \in E\}$$

$$S_1 = \{v \in u_r C v_s \mid u_s v \in E\}$$

$$R_2 = \{v \in u_s C u_r \mid u_r v \in E\}$$

$$S_2 = \{v \in u_s C v_r \mid u_s v^+ \in E\}$$

$$R_3 = \{v \in V \setminus V(C) \mid u_r v \in E\}$$

$$S_3 = \{v \in V \setminus V(C) \mid u_s v \in E\}$$

By lemma 2, $R_1 \cap S_1 = R_2 \cap S_2 = \emptyset$. Lemma 1 implies $R_3 \cap S_3 = \emptyset$. By lemma 2 and claim 1, either u_i or $v_i \notin R_1 \cup S_1 \cup R_2 \cup S_2$ for any $i \in \{1, 2, \dots, m\} \setminus \{r, s\}$. Thus,

$$\begin{aligned} d(x_r) + d(u_r) + d(u_s) &= d(x_r) + |R_1| + |R_2| \\ &\quad + |R_3| + |S_1| + |S_2| + |S_3| \\ &\leq (m + |V(H)| - 1) + (|V(C)| - (m - 2)) \\ &\quad + (|V| - |V(C)| - |V(H)|) = n + 1 \end{aligned}$$

Claim 2 $d(x_r) + d(u_r) + d(u_s) \leq n + 1$

Since S is a 2-independent set, we have $\sum(S) \geq n + 2$ by the condition of theorem 3. But this contradicts claim 2.

Proof of Theorem 2 By contradiction, suppose there exist non-Hamilton graphs which satisfy the condition of theorem 2. Let G be such a graph having the maximum number of edges. Take a longest cycle C such that $\Delta(V \setminus V(C))$ is as large as possible. Let x be the vertex such that $d(x) = \max\{d(v) \mid v \in V \setminus V(C)\}$. Set $N = N_C(x)$, then $N^+ = \{u_1, u_2, \dots, u_m\}$ and $\{x, u, v\}$ is a 2-inde-

pendent set for any $u, v \in N^+$. Set $F = \{x, u_1, u_2, \dots, u_m\}$. The condition of theorem 2 implies that.

Claim 1 $d(x) + d(u) + d(v) \geq n + k$ for any $u, v \in N^+$.

If $m = k$, then $d(u_1) + d(u_2) \geq n$. By the choice of G , $G + u_1 u_2$ is Hamiltonian. Hence G is Hamiltonian by Bondy's closure result.

Claim 2 $m \geq k + 1$ and $\alpha \geq m + 1$.

Claim 3 $(V \setminus V(C)) \cup N^+$ is independent.

In fact, $V \setminus V(C)$ is an independent set since C is a dominating cycle. Lemma 1 implies N^+ is independent. Hence, it suffices to show that no vertex in $V \setminus V(C)$ is adjacent to any vertex in N^+ . Suppose $x_1 (\neq x) \in V \setminus V(C)$ with $x_1 u_1 \in E$. Considering $S = \{x, x_1, u_k\}$. We have $\sum(S) \leq n - 1$. However, S is a 2-independent, the condition of theorem 2 implies $\sum(S) \geq n + k$, a contradiction. Thus claim 3 holds.

Claim 4 $3d(x) \geq n + k$

For any $u_i \in N^+$, if there exists $u_q (\neq u_i) \in N^+$ such that $u_q u_i^+ \in E$, then, considering the cycle $C' = u_q u_i^+ C v_q x v_i C u_q$ and vertex u_i , we have $d(u_i) \leq d(x)$ by the choice of x . Thus, if there exist such two vertices u_i, u_j , then $3d(x) \geq d(x) + d(u_i) + d(u_j) \geq n + k$, claim 4 holds. Hence, there exists at most such a vertex (If there exists, without loss of generality, assume that u_2 is such a vertex.). Consider u_1 and u_2 , $d(u_1) + d(u_2) = |R_1| + |R_2| + |S_1| + |S_2| \leq |V(C)| - d(x) + 2$, a contradiction. Hence claim 4 holds.

Let S be a vertex cut of cardinality k , and G_1, G_2, \dots, G_t be the components of $G - S$. Considering any pair of vertices $u_i, u_j \in N^+$, we have $d(u_i) + d(u_j) \geq n + k - m$. Since $|N(u_i) \cup N(u_j)| \leq |V(C)| - m < n - m$, we have $\lambda_{u_i u_j} > n + k - m - (n - m) = k$. It follows that any pair of vertices in N^+ cannot be in different components of $G - S$. Without loss of generality, assume $N^+ \subseteq V(G_1) \cup S$. By claim 2, we deduce that $|F \cap V(G_1)| \geq 1$. Let $B = V \setminus (S \cup V(G_1))$, $n_1 = |V(G_1)|$ and $n_2 = |B|$. Then $n = n_1 + n_2 + k$.

Claim 5 $|F \cap V(G_1)| \geq 2$

In fact, if $|F \cap V(G_1)| = 1$, say that $F \cap V(G_1) = \{u\}$, then clearly $u \neq x$, and $N^+ = S \cup \{u\}$, $N(u) \subseteq V(G_1)$, $x \in B$ and $m = k + 1$. Since $\delta \geq k \geq 2$, we have $n_1 \geq d(u) + 1 \geq \delta + 1 \geq k + 1$. Because $x \in B$ and $m = k + 1$, we have that $n_2 \geq m + 1 \geq k + 2$. It follows from claim 4 that $n = n_1 + n_2 + k \geq 3k + 3 = 3m \geq n + k$, a contradiction. Hence claim 5 holds.

Claim 6 $x \notin V(G_1)$

In fact, if $x \in V(G_1)$, consider $S' = \{x, w_1, y\}$ for any $y \in B$, the condition of theorem 2 implies $\sum(S') \geq n + k$. However, $n + k \leq d(x) + d(w_1) + d(y) \leq m + k + n_1 - (m + 1) + n_2 + k - 1 = n_1 + k + n_2 + k - 2 = n + k - 2$, a contradiction.

Claim 7 $x \in S$

If $x \notin S$, claim 6 implies $x \in B$. Then $N(x) = N_B(x) \cup N_S(x)$. Note that $N^+ \subseteq S \cup V(G_1)$, we have $m = |N_B(x) \cup N_S(x)| \leq |S| = k$. This contradicts claim 2.

Claim 8 $d(u, v) = 2$ for every pair of vertices $u, v \in N^+$. If not, then $d(u) + d(v) \leq |V(C)| - m$ for any pair of vertices $u, v \in N^+$. Consider a 2-independent set $\{x, u, v\}$, the condition of theorem 2 implies $d(x) + d(u) + d(v) \geq n + k$. However, $d(x) + d(u) + d(v) \leq m + n - 1 - m = n - 1$, a contradiction.

Claim 9 $d(w) > m$ for any $w \in F \cap V(G_1)$. If not, without loss of generality, assume that $d(w_1) \leq m$. Consider $S' = \{w_1, w_2, y\}$, where y is any vertex in B . Claim 8 implies that S' is a 2-independent set. From the condition of the theorem, we have that $d(w_1) + d(w_2) + d(y) \geq n + k$. That is, $n + k - d(y) \leq d(w_1) + d(w_2)$. Hence, on the one hand, we have $n + k - d(y) \geq n - n_2 + 1$, since $d(y) \leq n_2 + k - 1$. On the other hand, $d(w_1) + d(w_2) \leq m + d(w_2) \leq m + (n_1 + k - m - 1) = n_1 + k - 1$, since $d(w_1) \leq m$. The contradiction

implies claim 9 holds.

By the choice of x and claim 9, we have the following claim.

Claim 10 For any $u_i \in F \cap V(G_1)$, we have $u_j u_i^+ \notin E(G)$, $1 \leq j \neq i \leq m$.

Let $|N^+ \cap S| = s$. Note that $x \in S$ and $|S| = k$, we have $|F \cap V(G_1)| \geq m - s \geq m - k + 1$. when $s \geq 2$, take $u, v \in N^+ \cap S$; when $s = 1$, take $u \in N^+ \cap S$ and $v \in N^+ \cap V(G_1)$; When $s = 0$, take $u, v \in N^+ \cap V(G_1)$. Consider $S' = \{x, u, v\}$. Similarly to the proof of claim 1 in the proof of theorem 3, define R_1, R_2, S_1 and S_2 . Claim 10 implies that $u_i \notin R_1 \cup R_2 \cup S_1 \cup S_2$ for any $u_i \in F \cap V(G_1)$, and $d(u) + d(v) \leq |V(C)| - \min\{m - s, m - 1\}$. Hence, $n + k \leq d(x) + d(u) + d(v) \leq m - |V(C)| - \min\{m - s, m - 1\} \leq m + n - (m - k + 1) = n + k - 1$, a contradiction. Therefore the proof is completed.

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包含连通度的 Hamilton 图

周小跃

黄月年

(南京建筑工程学院, 南京 210009)

(金陵石油化工设计院, 南京 210042)

摘 要 设 G 是一个 2-连通简单图, 具有阶 n 和连通度 k . Bauer 等人已证明: 如果对任意三点独立集 $S = \{u, v, w\}$, 都有 $d(u) + d(v) + d(w) \geq n + k$, 则 G 是 Hamilton 图. 本文改进了这个结果. 如果一个独立集 S 中存在距离为 2 的 2 点, 则称 S 是一个 2-独立集. 本文证明了如下结果: 如果对任意 3 点 2-独立集 $S = \{u, v, w\}$, 都有 $d(u) + d(v) + d(w) \geq n + k$. 则 G 是 Hamilton 图. 这个结果意味我们仅需要检查所有 2-独立集是否满足条件.

关键词 连通度, 独立集, Hamilton 图

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