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Generalized Regular Ring

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Abstract: Using module class $C_R = \left\{M \middle| \forall x \in M, xR \otimes T = 0, \forall T \in \mathscr{T}\right\}$, we introduced the concepts of C_R-finitely generated module, C_R-finitely presented module and C_R-regular ring. We also discussed the criterion for C_R-regular ring, and the relations between C_R-regular ring and C_R- FP injective module.

Key words: C_R-finitely generated module, C_R- finitely presented module, C_R-regular ring, C_R-FP injective module

Regular ring is an important concept in ring theory. In this paper using module class $C_R = \left\{M \middle| \forall x \in M, xR \otimes T = 0, \forall T \in \mathscr{T}\right\}$, we define the concepts of C_R-finitely generated module and C_R-finitely presented module. Using these concepts, we approach the similar characterizations of regular in torsion theory.

Throughout this paper, all rings R are associative rings with identity and all modules M are unitary R-modules.

Definition 1 A module M_R is C_R-finitely generated module, if M_R is a finitely generated module and $M_R \in C_R$.

Definition 2 A module M_R is C_R-finitely presented if there exists an exact sequence $O \to K \to F \to M \to O$, where F is a finitely generated free module and K is a C_R-finitely generated module.

Definition 3 A module M_R is C_R-FP injective module, if every C_R-FP module F satisfies $\mathrm{Ext}^1(F, M) = 0$.

Definition 4 A ring R is C_R-regular ring if for each C_R-FG ideal I of R and $a \in I$, there exists $x \in R$ such that $a = axa$.

1 Main Result

Theorem 1 Let $(\mathscr{T}, \mathscr{R})$ be arbitrary torsion theory, $C_R = \left\{M \middle| \forall x \in M, xR \otimes T = 0, \forall T \in \mathscr{T}\right\}$, then (1)$\Rightarrow$(2)$\Rightarrow$(3). Where

1) The direct limit of C_R-FP injective module is C_R-FP injective module;

2) R is right C_R-coherent ring;

3) If X_R is C_R-FP injective module, then $X^+ = \mathrm{Hom}_Z(X, Q/Z)$ is C_R-flat module.

Proof (1)$\Rightarrow$(2) Let F be an arbitrary FP module, L be an arbitrary C_R-FG submodule of F. Suppose $(m_i)_I$ is a direct system. Since L is FG module[1], the morphism φ: $\varinjlim \mathrm{Hom}(L, M_i) \to \mathrm{Hom}(L, \varinjlim M_i)$ is monomornhism. Now we proof φ is epimorphism. For arbitrary morphism $\alpha: L \to \varinjlim M_i$, since $(M_i)_I$ is direct system, direct system of injective module $E(M_i) \supset M_i$. By 1), $\varinjlim E(M_i)$ is C_R-FP injective module. Since $\varinjlim$ is exact functor, there exists a monomorphism $\varphi: \varinjlim M_i \to \varinjlim E(M_i)$. For morphism $\varphi\alpha: L \to \varinjlim E(M_i)$, because $\varinjlim E(M_i)$ is C_R-FP injective module, by proposition 2.5[2], there exists a morphism $\beta: F \to \varinjlim E(M_i)$ with $\beta i = \varphi\alpha$, where i is morphism $L \to F$. So we have a commutative diagram with row exact (See Fig. 1).

$$
\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "i"] \arrow[d] & F \arrow[d] & & \\
0 \arrow[r] & \varinjlim M_i \arrow[r, "\varphi"] & \varinjlim E(M_i) \arrow[r, "h"] & \varinjlim E(M_i)/\varinjlim M_i \arrow[r] & 0
\end{tikzcd}
$$

Fig. 1 Exactness of the direct limit

Since F is FP module, then $\varinjlim \mathrm{Hom}(F, E(M_i)) \cong \mathrm{Hom}(F, \varinjlim E(M_i))$. For morphism β, there exists a unique morphism $\gamma: F \to E(M_i)$ with $L_i\gamma = \beta$, where L_i is embedding morphism of M_i to $\varinjlim M_i$. Since $h\varphi = 0$ and $\varphi\alpha = \beta i$, $h\beta i = 0$. Therefore compound morphism $L \to F \to E(M_j) \xrightarrow{\varphi'} E(M_j)/M_j$ is 0 under the direct limit. Since L is FG module, we can suppose compound morphism $\varphi'\gamma i = 0$ when j is big enough, we have a commutative diagram with row exact (See Fig. 2).

Since $\varphi'\gamma i = 0$, we have $\mathrm{im}\gamma i \subset \mathrm{Ker}\varphi' = \mathrm{im} i'$, then there exists $\delta: L \to M_j$ with $\gamma i = i'\delta$, so $l_i\gamma i = l_i i'\delta$. By $l_i\gamma = \beta$, $\beta i = \varphi\alpha$, then $\beta i = l_i i'\delta$, $\varphi\alpha =$

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$$\begin{tikzcd}
 & & l_i \arrow[dll] \arrow[d, "F"] & & \\
0 \arrow[r] & M_j \arrow[r, "i'"] & E(M_j) \arrow[r, "\varphi'"] & E(M_j)/M_j \arrow[r] & 0
\end{tikzcd}$$

Fig.2 Commutation diagram with row exact

$l_i i'\delta$. Because i', φ are embedding functors, $\alpha = l_i\delta$, $\mathrm{Hom}(L, \varinjlim M_i) \to \varinjlim \mathrm{Hom}(L, M_i)$ is epimorphism. So $\varinjlim \mathrm{Hom}(L, M_i) \to \mathrm{Hom}(L, \varinjlim M_i)$ is isomorphism, L is FP module, i.e. R is a right C_R-coherent ring.

(2)$\Rightarrow$(3) Let X_R be C_R-FP injective module and I be an arbitrary C_R-FG right ideal of R, then there exists an exact sequence $0 \to I \to R \to R/I \to 0$. Clearly R/I is C_R-FP module, so we have a long exact sequence: $\cdots \to \mathrm{Hom}(R, X) \to \mathrm{Hom}(I, X) \to \mathrm{Ext}^1(R/I, X_R) \to \cdots$ Since $\mathrm{Ext}^1(R/I, X_R) = 0$, there exists an exact sequence: $0 \to \mathrm{Hom}(R/I, X_R) \to \mathrm{Hom}(R, X_R) \to \mathrm{Hom}(I, X_R) \to 0$. Because Q/Z is an injective module, there exists an exact sequence: $0 \to \mathrm{Hom}(\mathrm{Hom}(I, X_R), Q/Z) \to \mathrm{Hom}(\mathrm{Hom}(R, X_R), Q/Z) \to \mathrm{Hom}(\mathrm{Hom}(R/I, X_R), Q/Z) \to 0$. Since I is C_R-FG ideal of R, R is C_R-coherent ring, then I is FP ideal. Clearly R, R/I are FP ideals, so $\mathrm{Hom}(\mathrm{Hom}(I, X), Q/Z) \cong \mathrm{Hom}(X, Q/Z) \otimes I$, $\mathrm{Hom}(\mathrm{Hom}(R, X), Q/Z) \cong \mathrm{Hom}(X, Q/Z) \otimes R$, $\mathrm{Hom}(\mathrm{Hom}(R/I, X), Q/Z) \cong \mathrm{Hom}(X, Q/Z) \otimes R/I$. Thus there exists an exact sequence : $0 \to I \otimes \mathrm{Hom}(X, Q/Z) \to R \otimes \mathrm{Hom}(X, Q/Z) \to R/I \otimes \mathrm{Hom}(X, Q/Z) \to 0$. $\mathrm{Hom}(X, Q/Z)$ is C_R-flat module by Ref.[2].

Remark Regular ring is C_R-regular ring.

Theorem 2 Let$(\mathscr{T}, \mathscr{R})$be any torsion theory, $C_R = \left\{ M \,\middle|\, \forall x \in M, xR \otimes T = 0, \forall T \in \mathscr{T} \right\}$. Then the following statements are equivalent for ring R.

(a) R is C_R-regular ring;

(b) If I is any C_R-FG right ideal of R, then for any $\forall a \in I$, aR is a right ideal generated by an idempotent element;

(c) Every C_R-FG right ideal of R is generated by an idempotent element;

(d) Every left R-module is C_R-flat.

Proof (a)$\Rightarrow$(b) Suppose I is an arbitrary C_R-FG ideal of R. $\forall a \in I$, since R is C_R-regular ring, there exists $x \in R$ with $a = axa$. Let $e = ax$, then $e^2 = ax \cdot ax = ax = e$, and $aR = axR = eR$.

(b)$\Rightarrow$(a) Let I be an arbitrary C_R-FG ideal of R, $a \in I$. By (b), aR is a right ideal generated by an idempotent element, namely there exists idempotent element e with $aR = eR$, so there exists $x \in R$ with $e = ax$. Since e is left identity of eR, then $ea = a$ and $a = ea = axa$, $\forall a \in I$. Therefore R is C_R-regular ring.

(a)$\Rightarrow$(c) Let I be an arbitrary C_R-FG ideal of R, $a_1, \cdots, a_n$ be generators of I, then $I = \sum_{i=1}^{n} a_i R$. By (a)$\Leftrightarrow$(b), $\forall a_i R$, there exists idempotent e_i with $a_i R = e_i R$, so $I = \sum_{i=1}^{n} e_i R$ and $e_i R \in C_R$. To proof right ideal I is generated by an idempotent, we only need prove if e, f are idempotents and belong to C_R-FG ideal I, the ideal $eR + fR$ of I is generated by an idempotent. First we have $eR + fR = eR + (f - ef)R$, e, f belong to I, so we have $f - ef \in I$. Since $I \in C_R$, there exists $x \in R$ with $f - ef = (f - ef)x(f - ef)$. Hence $f' = (f - ef)x$ is an idempotent and $ef' = 0$, so $eR + fR = eR + f'R$. Because $e = (e + f' - f'e)e$, $f' = (e + f' - f'e)f'$, we have $eR + fR = (e + f' - f'e)R$. That is $eR + fR$ be generated by $(e + f' - f'e)$, then I is generated by an idempotent.

(c)$\Rightarrow$(d) Let I be an arbitrary C_R-FG ideal of R. By (c), there exists an idempotent e with $I = eR$, then I is direct summand of R, so the exact sequence $0 \to I \to R \to R/I \to 0$ is split exact sequence, and $0 \to I \to R \to R/I \to 0$ is pure exact sequence. For an arbitrary module M, we have exact sequence: $0 \to I \otimes M \to R \otimes M \to R/I \otimes M \to 0$, so M is C_R-flat module. Since M is arbitrary, hence any R-module is C_R-flat module.

(d)$\Rightarrow$(a) Let I be an arbitrary C_R-right ideal of R and a is an arbitrary element of I. For each $ar \in aI$, we have $arR \otimes T = 0$, $\forall T \in \mathscr{T}$, ($ar \in I$), so $aI \in C_R$. Let $B = Ra$, then B is a left ideal of R. By (d), A/B is C_R-flat module. For canonical injection $i: aR \to R$, there exists a canonical morphism: $\varphi: aR \otimes R/B \to R \otimes R/B = R/B$, it is a monomorphism. Since $aR \otimes R/B \cong aR/(aR \cdot B)$(See Ref. [1]), canonical morphism $\psi: aR/(aR \cdot B) \to R/B$ is monomorphism, $\mathrm{Ker}\varphi = aR \cdot B$, i.e. $aR \cdot B = aR \cap B$. By $B = Ra$, we have $aR \cdot Ra = aR \cap Ra$, that is $aRa = aR \cap Ra$. Clearly $a \in aR \cap Ra$, so there exists $x \in R$ with $a = axa$, i.e. R is a C_R-regular ring.

Lemma 1 Let $(\mathscr{T}, \mathscr{R})$ be an arbitrary torsion theory, $C_R = \left\{ M \,\middle|\, \forall x \in M, xR \otimes T = 0, \forall T \in \mathscr{T} \right\}$, if any C_R-FG right ideal of R is direct summand of R, the R-module M is C_R-FP injective module $\Leftrightarrow \mathrm{Ext}^1(R/I, M) = 0$.

Proof $\Rightarrow$ Clearly. Because R/I is C_R-FP module.

$\Leftarrow$ Suppose $\mathrm{Ext}^1(R/I, M) = 0$ for all C_R-FG right ideal I. Let $f: N \to M$ be an arbitrary morphism, where N is a C_R-FG submodule of a finite generate free module F. Use mathematical induction for generator number of F.

When $n = 1, F = R$, clearly this proposition is identity.

Suppose the free module F' is generated by $k(k \leqslant n-1)$ elements, and the morphism of F' from C_R-FG submodule to M can all extend to the morphism of F' to M. If F is free module generating by n elements, then there exists $x \in F$ with $F = Rx \oplus V$, where V is free module generated by $k(k \leqslant n-1)$ elements. Suppose $I = \left\{ r \in R \,\middle|\, rx \in N + V \right\}$, then there exists isomorphism $I \cong N/(N \cap V)$. By Ref. [2], I is C_R-FG module. Because any C_R-FG of R is direct summand, so I is direct summand of R, and I is a projective module, i.e. $N/(N \cap R)$ is a projective module. So the exact sequence $0 \to N \cap V \to N \to N/(N \cap V) \to 0$ is split exact sequence and $N \cap V$ is direct summand of N, then $N \cap V$ is FG module. $\forall x \in V \cap N, x \in N$. Since $N \in C_R$, $xR \otimes T = 0, \forall T \in \mathscr{T}, N \cap V \in C_R$. By supposing, for morphism $f\big|_{N \cap V}: N \cap V \to M$ there exists a morphism $g: V \to A$ with $g\big|_{N \cap V} = f\big|_{N \cap V}$. Let morphism $\theta: N + V \to M$ with $\theta(n+v) = f(n) + g(v)$, clearly θ is extension of gand f. Let $\varphi: I \to M$ with $\varphi(r) = \theta(rx)$. Since $\mathrm{Ext}^1(R/I, M) = 0$, morphism $\mathrm{Hom}(R, M) \to \mathrm{Hom}(I, M)$ is epimorphism, and there exists morphism $\psi: R \to M$ with $\psi\big|_I = \varphi$. Let function $g: F \to M$ with $g(v + rx) = g(v) + \psi(r), v \in V, r \in R$, therefore $\forall y \in N$, we have $y = v_0 + r_0 x$, where v_0 and $r_0 x$ all belong to N and $v_0 \in N \cap V$, $g\big|_N(y) = g\big|_N(v_0 + r_0 x) = f(v_0) + \varphi(r_0) = f(v_0) + \theta(r_0 x) = f(v_0) + f(r_0 x) = f(v_0 + r_0 x) = f(y)(r_0 x \in N)$. So $r_0 \in I, \varphi(r_0) = \theta(r_0 x)$, then g is the extension of f. By Ref. [2], M is C_R-FP injective module.

Theorem 3 Let $(\mathscr{T}, \mathscr{R})$ be an arbitrary torsion theory, $C_R = \left\{ M \,\middle|\, \forall x \in M, xR \otimes T = 0, \forall T \in \mathscr{T} \right\}$. Then R is C_R-regular ring $\Leftrightarrow$ every right R-module is C_R-FP injective module.

Proof $\Rightarrow$By Ref. [2], any C_R-FG ideal I of R is generated by an idempotent element, i.e. $I = eR$. Therefore I is direct summand of R, i.e. the exact sequence $0 \to I \to R \to R/I \to 0$ is split exact sequence, then R/I is direct summand of R, i.e. R/I is projective module. For arbitrary R-module M, we have $\mathrm{Ext}_R^1(R/I, M) = 0$. By lemma 1, M is C_R-FP injective module.

$\Leftarrow$ Suppose all R-module are C_R-FP injective module, I is an arbitrary C_R-FG ideal of R. Then I as a right R-module is C_R-FP injective module. For C_R-FP module R/I we have $\mathrm{Ext}_R^1(R/I, I) = 0$, i.e. for exact sequence $0 \to I \to R \to R/I \to 0$, there exists exact sequence $0 \to \mathrm{Hom}(R/I, I) \to \mathrm{Hom}(R, I) \to \mathrm{Hom}(I, I) \to 0$. For morphism 1_I, there exists morphism $\alpha: R \to I$ with $\alpha\beta = 1_I$, i.e. $0 \to I \to R \to R/I \to 0$ is split exact sequence, where I is direct summand of R. Then there exists idempotent element e such that $I = Re$, so R is C_R-regular ring.

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广义正则环

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摘　要　利用模类 $C_R = \left\{ M \,\middle|\, \forall x \in M, xR \otimes T = 0, \forall T \in \mathscr{T} \right\}$，给出了 C_R -有限生成模、C_R -有限表示模、C_R -正则环的概念.研究了 C_R -正则环的判定条件及 C_R -正则环与 C_R - FP 内射模的关系.

关键词　C_R -有限生成模，C_R -有限表示模，C_R -正则环，C_R - FP 内射模

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