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On a Result of Niino and Ozawa*

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Abstract: In this paper, we study the deficient relation of some transcendental entire functions. If $f_j(z)(j=1,2,\cdots,p)$ be transcendental entire functions, and let $a_j(j=1,2,\cdots,p)$ be nonzero finite complex numbers. If $\sum_{j=1}^{p} a_j f_j(z)\equiv 1$, then $\sum_{j=1}^{p}\delta_{p-1}(0,f_j)\leqslant p-1$, where $\delta_{p-1}(0,f_j)=1-\overline{\lim_{r\to\infty}}\dfrac{N_{p-1}(r,1/f_j)}{T(r,f_j)}$ $(j=1,2,\cdots,p)$. The result improves a result of Niino and Ozawa. Meanwhile we give some applications of our result.

Key words: entire function, meromorphic function, deficiency

Let $f(z)$ be a nonconstant meromorphic function in the whole complex plane. We use the following standard notations of value distribution theory(see Refs.[1] and [2]).

$$T(r,f),\ m(r,f),\ N(r,f),\overline{N}(r,f),\cdots$$

We denote by $S(r,f)$ any function satisfying

$$S(r,f)=o\{T(r,f)\}$$

as $r\to+\infty$, possibly outside of a set with finite measure.

Let a be a finite complex number, k a positive integer. We denote by $N_{k)}\left(r,\dfrac{1}{f-a}\right)$ the counting function for zeros of $f(z)-a$ with multiplicity at most k, and by $\overline{N}_{k)}\left(r,\dfrac{1}{f-a}\right)$ the corresponding one for which multiplicity is not counted. Let $N_{(k}\left(r,\dfrac{1}{f-a}\right)$ be the counting function for zeros of $f(z)-a$ with multiplicity at least k and $\overline{N}_{(k}\left(r,\dfrac{1}{f-a}\right)$ the corresponding one for which multiplicity is not counted. Set $N_k\left(r,\dfrac{1}{f-a}\right)=\overline{N}\left(r,\dfrac{1}{f-a}\right)+\overline{N}_{(2}\left(r,\dfrac{1}{f-a}\right)+\cdots+\overline{N}_{(k}\left(r,\dfrac{1}{f-a}\right)$. We define

$$\delta_k(a,f)=1-\overline{\lim_{r\to\infty}}\frac{N_k\left(r,\dfrac{1}{f-a}\right)}{T(r,f)}$$

In Ref. [3], Niino and Ozawa proved the following result.

Theorem A Let $f_j(z)(j=1,2,\cdots,p)$ be transcendental entire functions, and let $a_j(j=1,2,\cdots,p)$ be nonzero finite complex numbers. If $\sum_{j=1}^{p}a_jf_j(z)\equiv 1$, then

$$\sum_{j=1}^{p}\delta(0,f_j)\leqslant p-1$$

where $\delta(0,f_j)=1-\overline{\lim_{r\to\infty}}\dfrac{N\left(r,\dfrac{1}{f_j}\right)}{T(r,f_j)}$, $j=1,2,\cdots,p$.

In this paper, we improve theorem A as follows.

Theorem 1 Let $f_j(z)$ $(j=1,2,\cdots,p)$ be transcendental entire functions, and let $a_j(j=1,2,\cdots,p)$ be nonzero finite complex numbers. If $\sum_{j=1}^{p}a_jf_j(z)\equiv 1$, then

$$\sum_{j=1}^{p}\delta_{p-1}(0,f_j)\leqslant p-1$$

where $\delta_{p-1}(0,f_j)=1-\overline{\lim_{r\to\infty}}\dfrac{N_{p-1}\left(r,\dfrac{1}{f_j}\right)}{T(r,f_j)}$, $j=1,2,\cdots,p$.

Remark 1 For any positive integer k, we have $\delta(0,f_j)\leqslant\delta_{p-1}(0,f_j)$.

1 Some Lemmas

For the proof of theorem 1 we need the following lemmas.

Lemma 1[4] Let $f_j(z)$ $(j=1,2,\cdots,p)$ be linearly independent meromorphic functions, p a positive integer. If

$$\sum_{j=1}^{p}f_j(z)\equiv 1$$

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then for $1 \leqslant j \leqslant p$

$$T(r,f_j) \leqslant \sum_{i=1}^{p} N\left(r,\frac{1}{f_i}\right) + N(r,f_j) + N(r,W) - \sum_{i=1}^{p} N(r,f_i) - N\left(r,\frac{1}{W}\right) + S(r)$$

where $W(f_1,f_2,\cdots,f_p)$ is the Wronskian determinent of $f_j(z)$ $(j = 1,2,\cdots,p)$.

$$S(r) = o(T(r)) \qquad r \to \infty, r \notin E$$

here

$$T(r) = \max_{1\leqslant j\leqslant p}\{T(r,f_j)\}$$

and E is a set of finite measure.

By lemma 1 we can easily obtain lemma 2.

Lemma 2 Let $f_j(z)$ $(j = 1,2,\cdots p)$ be linearly independent transcendental entire functions. If

$$f_1(z) + f_2(z) + \cdots + f_p(z) \equiv 1$$

then for $1 \leqslant j \leqslant p$

$$T(r,f_j) \leqslant \sum_{i=1}^{p} N_{p-1}\left(r,\frac{1}{f_i}\right) + S(r)$$

Here $S(r)$ is the same as in lemma 1.

2 Proof of Theorem 1

Case 1 $f_1,f_2,\cdots,f_p$ are linearly independent functions. Then by lemma 2 we have

$$T(r,f_j) \leqslant \sum_{i=1}^{p} N_{p-1}\left(r,\frac{1}{f_i}\right) + S(r) \leqslant \sum_{i=1}^{p}[1 - \delta_{p-1}(0,f_i)]T(r,f_i) + S(r)$$

Thus we obtain

$$T(r) \leqslant \sum_{i=1}^{p}[1 - \delta_{p-1}(0,f_i)]T(r) + S(r)$$

That is

$$\left[\sum_{i=1}^{p}\delta_{p-1}(0,f_i) - (p-1)\right]T(r) \leqslant S(r)$$

Hence we get

$$\sum_{i=1}^{p}\delta_{p-1}(0,f_i) \leqslant p - 1 \tag{1}$$

Case 2 f_1, $f_2,\cdots,f_p$ are linearly dependent functions. Without loss of generality, we assume that $f_1,f_2,\cdots,f_q$ are linearly independent functions and that $f_{q+1},f_{q+2},\cdots,f_p$ can be linearly expressed as $f_1,f_2,\cdots,f_q$. Thus there exist constants c_1, $c_2,\cdots,c_q$ such that

$$\sum_{i=1}^{q} c_i f_i \equiv 1$$

If $c_i \neq 0(i = 1, 2,\cdots,q)$, then by the same argument as do in case 1 we obtain

$$\sum_{i=1}^{q}\delta_{q-1}(0,f_i) \leqslant q - 1$$

Obviously, we have

$$\delta_{q-1}(0, f_i) \geqslant \delta_{p-1}(0, f_i) \qquad i = 1,2,\cdots,q$$

Hence we obtain

$$\sum_{i=1}^{p}\delta_{p-1}(0,f_i) \leqslant \sum_{i=1}^{p}\delta_{p-1}(0,f_i) + \sum_{i=q+1}^{q}\delta_{p-1}(0,f_i) \leqslant q - 1 + p - q = p - 1 \tag{2}$$

If there exists $c_i = 0$, without loss of generality we assume that $c_i \neq 0(i = 1,2,\cdots,r)$ and $c_{r+1} = 0,\cdots$, $c_q = 0$, then we can similarly prove

$$\sum_{i=1}^{p}\delta_{p-1}(0,f_i) \leqslant p - 1$$

The proof of the theorem is complete.

3 An Application of Theorem 1

By theorem 1, we obtain the following results.

Theorem 2 Let $k,n_1,n_2,\cdots,n_k$ be positive integers satisfying

$$\sum_{i=1}^{k}\frac{1}{n_i} < \frac{1}{k-1} \tag{3}$$

Then there don't exist transcendental entire functions $f_1(z),f_2(z),\cdots, f_k(z)$ such that

$$f_1^{n_1}(z) + f_2^{n_2}(z) + \cdots + f_k^{n_k}(z) \equiv 1 \tag{4}$$

Proof Suppose that there exist transcendental entire functions $f_1(z),f_2(z),\cdots, f_k(z)$ satisfying

$$f_1^{n_1}(z) + f_2^{n_2}(z) + \cdots + f_k^{n_k}(z) \equiv 1$$

Then by theorem 1 we get

$$\sum_{i=1}^{k}\delta_{k-1}(0, f_i^{n_i}) \leqslant k - 1 \tag{5}$$

On the other hand, we have

$$\delta_{k-1}(0, f_i^{n_i}) = 1 - \overline{\lim_{r\to\infty}}\frac{N_{k-1}\left(r,\frac{1}{f_i^{n_i}}\right)}{T(r,f_i^{n_i})} \geqslant 1 - \overline{\lim_{r\to\infty}}\frac{(k-1)N\left(r,\frac{1}{f_i}\right)}{n_iT(r,f_i)} \geqslant 1 - \frac{k-1}{n_i}$$

Thus we get

$$\sum_{i=1}^{k}\delta_{k-1}(0,f_i^{n_i}) \geqslant k - \sum_{i=1}^{k}\frac{k-1}{n_i} \tag{6}$$

Hence by (5) and (6) we have

$$\sum_{i=1}^{k}\frac{k-1}{n_i} \geqslant 1$$

which contradicts (3). The proof of theorem 2 is complete.

From theorem 2, we get the following corbllary.

Corollary 1 Let $n \geqslant 7$ be a positive integer. Then there don't exist three transcendental entire functions $f(z)$, $g(z),h(z)$ such that

$$f^n(z) + g^n(z) + h^n(z) \equiv 1$$

Corollary 2 Let $n \geqslant 3$ be a positive integer.

Then there don't exist two transcendental entire functions $f(z)$, $g(z)$, such that

$$f^n(z) + g^n(z) \equiv 1$$

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关于 Niino 和 Ozawa 的一个结果

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摘　要　讨论了一些超越整函数亏量间的关系.如果 $f_1(z), f_2(z), \cdots, f_p(z)$ 都是超越整函数, $a_1, a_2, \cdots, a_p$ 都是非零有限复数,并且 $\sum_{j=1}^{p} a_j f_j \equiv 1$,则 $\sum_{j=1}^{P} \delta(0, f_j) \leqslant p - 1$.这里 $\delta(0, f_j) = 1 - \overline{\lim_{r \to +\infty}} \dfrac{N\left(r, \dfrac{1}{f_j}\right)}{T(r, f_j)}$ $(j = 1, 2, \cdots, p)$.这个结果改进了 Niino 和 Ozawa 的一个结果.同时本文又给出了这个结果的一些应用.

关键词　整函数,亚纯函数,亏量

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