

Blow-Up Rate for a Semilinear Parabolic System^{*}

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Abstract: This paper deals with the blow-up rate of positive solution for a semilinear parabolic system coupled in the equations and a boundary condition. The upper and lower bounds of blow-up rates are obtained.

Key words: parabolic systems, nonlinear boundary conditions, blow-up rates, upper and lower bounds

1 Main Results

Let p, q and m be positive constants. In this paper, we study the blow-up rate estimate of positive solution to the following semilinear parabolic system in half plane with nonlinear boundary condition:

$$\left. \begin{aligned} u_t &= u_{xx} + v^p, \quad v_t = v_{xx} + u^q & x > 0, \quad t > 0 \\ -u_x &= v^m, \quad -v_x = 0 & x = 0, \quad t > 0 \\ u(x, 0) &= u_0(x), \quad v(x, 0) = v_0(x) & x \geq 0 \end{aligned} \right\} \quad (1)$$

where the initial data $u_0(x)$ and $v_0(x)$ are bounded nonnegative and nontrivial C^1 functions and satisfy

$$\left. \begin{aligned} -u_{0x} &= \{v_0\}^m, \quad -v_{0x} = 0 & \text{for } x = 0 \\ \lim_{x \rightarrow \infty} u_0(x) &= 0, \quad \lim_{x \rightarrow \infty} v_0(x) = 0 \end{aligned} \right\} \quad (2)$$

Using the comparison principle and the results of Refs. [1, 2], we know that if

$$\max\{pq - 1, mq - 1\} > 0 \quad (3)$$

then the solution (u, v) of (1) blows up in finite time for the suitable “large” initial data. Throughout this paper we assume that (3) holds and the solution (u, v) of (1) blows up in finite time T . It is obvious that u and v blow up simultaneously. By (3) it follows that

$$\left. \begin{aligned} p &\geq (2mq + 2m - 1)/(2 + q) \Rightarrow pq > 1 \\ p &\leq (2mq + 2m - 1)/(2 + q) \Rightarrow mq > 1 \end{aligned} \right\} \quad (4)$$

There are many results on the blow-up rates for semilinear parabolic systems with nonlinear boundary conditions (see Refs. [3 – 6] and the references therein). The main goal of this note is to study the blow-up rate estimate of solution to (1), and find out the whole effects of the reaction terms v^p , u^q and the nonlinear boundary condition u^m on the blow-up rate. Explicitly, our main results of the present paper read as follows.

Theorem 1 Assume that the initial data (u_0, v_0) satisfies (2).

(i) There exists positive constant c , such that if $p \geq (2mq + 2m - 1)/(2 + q)$ then

$$\left. \begin{aligned} \max_{0 \leq \tau \leq t} \sup_{x \geq 0} u(x, \tau) &\geq c(T - t)^{-(p+1)/(pq-1)} \\ \max_{0 \leq \tau \leq t} \sup_{x \geq 0} v(x, \tau) &\geq c(T - t)^{-(q+1)/(pq-1)} \end{aligned} \right\} \quad (5)$$

while, if $p \leq (2mq + 2m - 1)/(2 + q)$ then

$$\left. \begin{aligned} \max_{0 \leq \tau \leq t} \sup_{x \geq 0} u(x, \tau) &\geq c(T - t)^{-(1+2m)/[2(mq-1)]} \\ \max_{0 \leq \tau \leq t} \sup_{x \geq 0} v(x, \tau) &\geq c(T - t)^{-(2+q)/[2(mq-1)]} \end{aligned} \right\} \quad (6)$$

(ii) When $p \geq (2mq + 2m - 1)/(2 + q)$, we assume $\max\{(p + 1)/(pq - 1), (q + 1)/(pq - 1)\} \geq 1/2$; when $p \leq (2mq + 2m - 1)/(2 + q)$, we assume that the initial data (u_0, v_0) satisfies $u'_0(x) \leq 0$, $v'_0(x) \leq 0$, and the parameters m and q satisfy $\max\{(2 + q)/(mq - 1), (1 + 2m)/(mq - 1)\} > 1$, or $\max\{(2 + q)/(mq - 1), (1 + 2m)/(mq - 1)\} = 1$, and $q, m \geq 1$.

Then there exists positive constant C , such that if $p \leq (2mq + 2m - 1)/(2 + q)$ then

$$\left. \begin{aligned} \max_{0 \leq \tau \leq t} \sup_{x \geq 0} u(x, \tau) &\leq C(T - t)^{-(1+2m)/[2(mq-1)]} \\ \max_{0 \leq \tau \leq t} \sup_{x \geq 0} v(x, \tau) &\leq C(T - t)^{-(2+q)/[2(mq-1)]} \end{aligned} \right\} \quad (7)$$

while, if $p \geq (2mq + 2m - 1)/(2 + q)$ then

$$\left. \begin{aligned} \max_{0 \leq \tau \leq t} \sup_{x \geq 0} u(x, \tau) &\leq C(T - t)^{-(1+p)/(pq-1)} \\ \max_{0 \leq \tau \leq t} \sup_{x \geq 0} v(x, \tau) &\leq C(T - t)^{-(1+q)/(pq-1)} \end{aligned} \right\} \quad (8)$$

Remark If $p = (2mq + 2m - 1)/(2 + q)$, then $(1 + 2m)/[2(mq - 1)] = (1 + p)/(pq - 1)$ and $(2 + q)/[2(mq - 1)] = (1 + q)/(pq - 1)$.

2 Proof of Theorem 1

By the assumption of theorem 1, it follows from the classical results that the solution (u, v) of (1) satisfies $u(x, t) \geq 0$, $v(x, t) \geq 0$ and $\lim_{x \rightarrow \infty} u(x, t) = 0$, $\lim_{x \rightarrow \infty} v(x, t) = 0$. Therefore, $u(x, t)$ and $v(x, t)$ obtain their maximums in the interior of $[0, +\infty)$.

Denote $f(t) = \max_{0 \leq \tau \leq t} \sup_{x \geq 0} u(x, \tau)$, $g(t) = \max_{0 \leq \tau \leq t} \sup_{x \geq 0} v(x, \tau)$, then $f(t)$ and $g(t)$ are the non-decreasing functions in t . We first use the ideas of Ref. [7] to prove the following lemma.

Lemma 1 Let α and β be positive constants and satisfy

$$2 + \alpha - p\beta = 0, \quad 2 + \beta - q\alpha = 0 \quad (9)$$

Then there exists a positive constant ε such that, $\forall t \in [T/2, T)$,

$$\varepsilon g^{1/\beta}(t) \leq f^{1/\alpha}(t), \quad \varepsilon f^{1/\alpha}(t) \leq g^{1/\beta}(t) \quad (10)$$

Proof On the contrary we assume that the first inequality of (10) is not true, then there exists a sequence $\{t_n\}$ with $t_n \rightarrow T^-$ such that $g^{-1/\beta}(t_n) f^{1/\alpha}(t_n) \rightarrow 0$ as $n \rightarrow \infty$. For each t_n , choose $(\hat{x}_n, \hat{t}_n) \in R^+ \times (0, t_n]$ such that $v(\hat{x}_n, \hat{t}_n) = g(t_n)$. Since $g(t_n) \rightarrow \infty$, it follows that $\hat{t}_n \rightarrow T^-$. Let

$$\left. \begin{aligned} \lambda_n &= g^{-1/\beta}(t_n) \\ \varphi_n(y, s) &= \lambda_n^\alpha u(\lambda_n y + \hat{x}_n, \lambda_n^2 s + \hat{t}_n) \\ \psi_n(y, s) &= \lambda_n^\beta v(\lambda_n y + \hat{x}_n, \lambda_n^2 s + \hat{t}_n) \\ (y, s) &\in J_n \times I_n(T) \end{aligned} \right\} \quad (11)$$

where $J_n = [-\lambda_n^{-1} \hat{x}_n, +\infty)$, $I_n(t) = (-\lambda_n^{-2} \hat{t}_n, \lambda_n^{-2}(t - \hat{t}_n))$. By the direct computations, we see that φ_n and ψ_n satisfy

$$\left. \begin{aligned} (\varphi_n)_s &= \{(\varphi_n)\}_{yy} + \lambda_n^{2+\alpha-\beta\beta} \psi_n^p \\ (\psi_n)_s &= \{(\psi_n)\}_{yy} + \lambda_n^{2+\beta-q\alpha} \varphi_n^q \\ -\{(\varphi_n)\}_y(-\lambda_n^{-1} \hat{x}_n, s) &= \lambda^{1+\alpha-m\beta} \psi_n^m(-\lambda_n^{-1} \hat{x}_n, s) \\ -\{(\psi_n)\}_y(-\lambda_n^{-1} \hat{x}_n, s) &= 0 \end{aligned} \right\} \quad (12)$$

where $y \in J_n, s \in I_n(T)$ and

$$\left. \begin{aligned} \psi_n(0, 0) &= 1, \quad 0 \leq \psi_n(y, s) \leq 1 \\ 0 \leq \varphi_n(y, s) &\leq \lambda_n^\alpha f(t_n) = g^{-\alpha/\beta}(t_n) f(t_n) \end{aligned} \right\} \quad (13)$$

where $y \in J_n, s \in (-\lambda_n^{-2} \hat{t}_n, 0]$.

Using Eqs. (9) and (13) and $\lambda_n \rightarrow 0$, we know that the nonlinear terms in (12) are all uniformly bounded. For any $K > 0$, in view of (12) and the Schauder's estimate, we obtain

$$\|(\varphi_n, \psi_n)\|_{C^{2+\mu, 1+\mu/2}(\{|J_n \cap \{|y| \leq K\} \times [-K, 0]\})} \leq C_K$$

where the constant C_K is independent of n . It follows that there exist subsequences of $\{(\varphi_n, \psi_n)\}$ and $\{\hat{x}_n\}$, which denoted also by $\{(\varphi_n, \psi_n)\}$ and $\{\hat{x}_n\}$ respectively, nonnegative functions φ and ψ , and $a \geq -\infty$ such that $-\lambda_n^{-1} \hat{x}_n \rightarrow a$ and $(\varphi_n, \psi_n) \rightarrow (\varphi, \psi)$ locally uniformly on $(y, s) \in (a, +\infty) \times (-\infty, 0]$. Moreover, for $y \in (a, +\infty)$, $s \in (-\infty, 0]$, (φ, ψ) satisfies

$$\varphi_s = \varphi_{yy} + \varphi^p, \quad \psi_s = \psi_{yy} + \varphi^q$$

$$\psi(0, 0) = 1, \quad 0 \leq \psi \leq 1, \quad \varphi \equiv 0$$

It is a contradiction.

By the similar way, we can prove the second inequality of (10).

If $f(t) = \max_{0 \leq \tau \leq t} u(0, \tau)$, $g(t) = \max_{0 \leq \tau \leq t} v(0, \tau)$, we can prove the following similarly.

Lemma 2 Let α and β be positive constants and satisfy

$$2 + \alpha - p\beta \geq 0, \quad 2 + \beta - q\alpha = 1 + \alpha - m\beta = 0 \quad (14)$$

Then there exists a positive constant ε such that (10) holds.

Recall that the Green's function $G(x, y, t)$ for the heat equation in R^+ satisfying $\frac{\partial G}{\partial y} = 0$ at $y = 0$ is given by

$$G(x, y, t) = (4\pi t)^{-1/2} \left(\exp\left(-\frac{(x-y)^2}{4t}\right) + \exp\left(-\frac{(x+y)^2}{4t}\right) \right)$$

For any $0 \leq z < t < T$, we have Green's identity (See Ref. [4]),

$$\begin{aligned} u(x, t) &= \int_0^{+\infty} G(x, y, t) u(y, z) dy + \int_z^t \int_0^{+\infty} G(x, y, t-\eta) v^p(y, \eta) dy d\eta + \int_z^t G(x, 0, t-\eta) v^m(0, \eta) dy d\eta \\ v(x, t) &= \int_0^{+\infty} G(x, y, t) v(y, z) dy + \int_z^t \int_0^{+\infty} G(x, y, t-\eta) u^q(y, \eta) dy d\eta \end{aligned}$$

For simplicity we define positive constants α and β as follows:

$$\left. \begin{aligned} \alpha &= \frac{2(p+1)}{pq-1}, \quad \beta = \frac{2(q+1)}{pq-1} \quad \text{if } p \geq \frac{2mq+2m-1}{2+q} \\ \alpha &= \frac{2m+1}{mq-1}, \quad \beta = \frac{2+q}{mq-1} \quad \text{if } p \leq \frac{2mq+2m-1}{2+q} \end{aligned} \right\} \quad (16)$$

2.1 Proof of the lower bounds

To prove the lower bounds, we first prove a lemma.

Lemma 3 Let the positive constants α and β be given by (16). Then for $T/2 < t < T$, we have

$$\begin{aligned} \text{(A) If } p \geq 2m - \alpha/\beta, \text{ then} \\ f(z) &\geq c(T-z)^{-1/(p\beta/a-1)} \\ g(z) &\geq c(T-z)^{-1/(p-\alpha/\beta)} \end{aligned} \quad (17)$$

$$\begin{aligned} \text{(B) If } p \leq 2m - \alpha/\beta, \text{ then} \\ f(z) &\geq c(T-z)^{-1/[2(m\beta/a-1)]} \\ g(z) &\geq c(T-z)^{-1/[2(m-\alpha/\beta)]} \end{aligned} \quad (18)$$

Proof By the expressions of α and β we see that if $p \geq (2mq + 2m - 1)/(2 + q)$, then (9) holds, and if $p \leq (2mq + 2m - 1)/(2 + q)$, then (14) holds. Hence we have (10). In view of (15) and the first inequality of (10), we have

$$\begin{aligned} f(t) &\leq f(z) + \int_z^t g^p(\eta) d\eta + \\ &g^m(t) \int_z^t \pi^{-1/2}(t - \eta)^{-1/2} d\eta \leq \\ &f(z) + (T - z)g^p(t) + \\ &C(T - z)^{1/2}g^m(t) \leq \\ &f(z) + C(T - z)f^{p\beta/\alpha}(t) + \\ &C(T - z)^{1/2}f^{m\beta/\alpha}(t) \end{aligned} \quad (19)$$

By our assumption, $f(t) \rightarrow +\infty$ as $t \rightarrow T^-$. For any $z \in (T/2, T)$, one can choose $t: z < t < T$ such that $f(t) = 2f(z)$. Without loss of generality we assume that $f(t), g(t) > 1$ for $T/2 < z < T$.

(A) If $p \geq 2m - \alpha/\beta$, i.e. $p\beta/\alpha \geq 2m\beta/\alpha - 1$, then by (19) we have

$$\begin{aligned} f(z) &\leq C(T - z)f^{p\beta/\alpha}(z) + \\ &C(T - z)^{1/2}f^{p\beta/(2\alpha)}(z)f^{1/2}(z) \leq \\ &C(T - z)f^{p\beta/\alpha}(z) + \varepsilon f(z) + \\ &C(\varepsilon)(T - z)f^{p\beta/\alpha}(z) \leq \\ &C(T - z)f^{p\beta/\alpha}(z) \quad T/2 < z < T \end{aligned}$$

This implies the first inequality of (17). By (10), we obtain the second inequality of (17).

(B) If $p \leq 2m - \alpha/\beta$, i.e. $p\beta/\alpha \leq 2m\beta/\alpha - 1$, then by (19) we have

$$\begin{aligned} f(z) &\leq C(T - z)f^{2m\beta/\alpha - 1} + \\ &C(T - z)^{1/2}f^{(2m\beta/\alpha - 1)/2}(z)f^{1/2}(z) \leq \\ &C(T - z)f^{2m\beta/\alpha - 1} + \varepsilon f(z) + \\ &C(\varepsilon)(T - z)f^{2m\beta/\alpha - 1}(z) \leq \\ &C(T - z)f^{2m\beta/\alpha - 1}(z) \quad T/2 < z < T \end{aligned}$$

Similar to the case (A) we see that (18) holds. The proof is completed.

In the following we give the proof of the lower bounds.

1) If $p \geq (2mq + 2m - 1)/(2 + q)$, then $pq > 1$ by (4). Using (16) and the direct calculation we have $2 + \alpha - p\beta = 2 + \beta - q\alpha = 0$, $1 + \alpha - m\beta \geq 0$, $p\beta/\alpha + 1 - 2m\beta/\alpha \geq 0$. This shows that the conditions of lemma 1 hold and $p \geq 2m - \alpha/\beta$. Therefore, (17) holds. The direct computations give $p\beta/\alpha - 1 = (pq - 1)/(p + 1)$, $p - \alpha/\beta = (pq - 1)/(q + 1)$. Hence (5) holds.

2) If $p \leq (2mq + 2m - 1)/(2 + q)$, then $pq > 1$ by (4). In view of (16) and by the direct calculation we see that $2 + \beta - q\alpha = 1 + \alpha - m\beta = 0$, $2 + \alpha - p\beta \geq 0$, $p\beta/\alpha + 1 - 2m\beta/\alpha \leq 0$. This shows that the conditions of lemma 2 hold and $p \leq 2m - \alpha/\beta$.

Therefore, (18) holds, and consequently (6) holds.

2.2 Proof of the upper bounds

Since $g(t)$ is continuous, nondecreasing and $\lim_{t \rightarrow T^-} g(t) = \infty$. For any $t_0 \in (0, T)$, we define

$$t_0^+ = t^+(t_0) = \max\{t \in (t_0, T) \mid g(t) = 2g(t_0)\}$$

Choose $\lambda_0 = \lambda(t_0)$ as in (11), we assert that

$$\lambda^{-2}(t_0)(t_0^+ - t_0) \leq M \quad \forall t_0 \in (T/2, T) \quad (20)$$

where M is a positive constant which does not depend on t_0 . If (20) were false, then there would exist a sequence $t_n \rightarrow T^-$ such that $\lambda_n^{-2}(t_n^+ - t_n) \rightarrow \infty$, where $\lambda_n = \lambda(t_n)$ and $t_n^+ = t^+(t_n)$. For each t_n choose (\hat{x}_n, \hat{t}_n) such that $v(\hat{x}_n, \hat{t}_n) = g(t_n)$.

1) If $p \geq (2mq + 2m - 1)/(2 + q)$ then $pq > 1$. By direct calculation, we know that the conditions of lemma 1 hold. We rescale (u, v) around (\hat{x}_n, \hat{t}_n) as in (11), and then obtain a solution of (12) in $J_n \times I_n(T)$ such that $\psi_n(0, 0) = 1$. From (10) and the definition of t_n^+ we have

$$0 \leq \psi_n \leq \lambda_n^\beta g(t_n^+) = 2$$

$$0 \leq \varphi_n \leq \lambda_n^\alpha f(t_n^+) \leq \lambda_n^\alpha \varepsilon^{-\alpha} g^{a/\beta}(t_n^+) = 2^{a/\beta} \varepsilon^{-\alpha}$$

where $y \in J_n$, $s \in I_n(t_n^+)$.

Same as the proof of lemma 1, there exist $C^{2,1}$ functions $\varphi(y, s)$ and $\psi(y, s)$, which satisfy

$$\begin{cases} \varphi_s = \varphi_{yy} + \psi^p, & \psi_s = \psi_{yy} + \varphi^q \\ y \in R & s \in (-\infty, +\infty) \\ \varphi_0(y) \geq 0, & \psi_0(y) \geq 0 \quad y \in R \end{cases}$$

Since $\max\{(p + 1)/(pq - 1), (q + 1)/(pq - 1)\} \geq 1/2$, the results of Ref. [1] imply that (φ, ψ) blows up in finite time. It is a contradiction. So, (20) holds.

2) If $p \leq (2mq + 2m - 1)/(2 + q)$ then $pq > 1$. By the assumption on initial data we know that the solution of (1) satisfies $u_x \leq 0, v_x \leq 0$. Hence, $f(t) = \max_{0 \leq \tau \leq t} u(0, \tau)$, $g(t) = \max_{0 \leq \tau \leq t} v(0, \tau)$. By direct calculation we know that the conditions of lemma 2 hold. We rescale (u, v) around $(0, \hat{t})$ as in (11), and obtain a solution of (12) (in there \hat{x}_n is replaced by 0) such that $\psi_n(0, 0) = 1$, and

$$0 \leq \psi_n \leq \lambda_n^\beta g(t_n^+) = 2\lambda_n^\beta g(t_n) = 2$$

$$0 \leq \varphi_n \leq \lambda_n^\alpha f(t_n^+) \leq \lambda_n^\alpha \varepsilon^{-\alpha} (g(t_n^+))^{a/\beta} = 2^{a/\beta} \varepsilon^{-\alpha}$$

where $y \in R^+$, $s \in I_n(t_n^+)$. Same as the proof of lemma 1, there exist $C^{2,1}$ functions $\varphi(y, s)$ and $\psi(y, s)$, which satisfy

$$\begin{cases} \varphi_s = \varphi_{yy} + \delta_1 \psi^p, & \psi_s = \psi_{yy} + \varphi^q \\ -\varphi_y(0, s) = \psi^m(0, s), & -\psi_y(0, s) = 0 \end{cases}$$

where $y \in R^+$, $s \in (-\infty, +\infty)$, δ_1 satisfies $0 \leq \delta_1 \leq 1$. Recalling the assumptions of (ii) of theorem 1, using the results of Ref. [2] and the comparison principle, we know that (φ, ψ) blows up in finite time. It is a contradiction. Hence, (20) holds.

Next we use an idea from Ref. [7] to give the estimate of the upper bound. From (11) and (20), it follows that

$$t_0^+ - t_0 \leq M g^{-2/\beta}(t_0) \quad \forall t_0 \in (T/2, T)$$

Fix $t_0 \in (T/2, T)$ and denote $t_1 = t_0^+$, $t_2 = t_1^+$, $t_3 = t_2^+$, \dots . Then

$$t_{j+1} - t_j \leq M g^{-2/\beta}(t_j), \quad g(t_{j+1}) = 2g(t_j) \\ j = 0, 1, 2, \dots$$

Consequently

$$T - t_0 = \sum_{j=0}^{\infty} (t_{j+1} - t_j) \leq M \sum_{j=0}^{\infty} g^{-2/\beta}(t_j) = \\ M g^{-2/\beta}(t_0) \sum_{j=0}^{\infty} 4^{-j/\beta}$$

Hence

$$v(x, t_0) \leq g(t_0) \leq C(T - t_0)^{-\beta/2} \quad t_0 \in (T/2, T)$$

where $C = \left(M \sum_{j=0}^{\infty} 4^{-j/\beta}\right)^{\beta/2} = \left(M \sum_{j=0}^{\infty} 4^{-j/\beta}\right)^{\beta/2}$. With

this conclusion and (10), we can get (7) and (8).

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一个抛物型方程组爆破解的速率估计

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摘 要 研究一个通过方程和非线性边界条件耦合的抛物型方程组爆破解的速率估计,给出了非线性反应项和非线性边界条件对于爆破速率的影响的精确刻划.

关键词 抛物型方程, 非线性边界条件, 爆破速率

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