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Structural Analogy for Dynamic Analysis of Liquid Sloshing

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Abstract: The analogy between the wave equation of liquid and the Navier equations of structural elasticity is examined in detail. By introducing appropriate parameters, the structural counterpart of the liquid sloshing model can be easily built. Therefore, the dynamic analysis of liquid sloshing can be reduced to that of structural elasticity, and the existing FEM structural analysis computer programs can be applied to liquid sloshing analysis without any modification. The present method also reveals the internal relationship between liquid sloshing and structural vibration. The effectiveness and reliability of the method is illustrated by the numerical example.

Key words: liquid sloshing, dynamic analysis, liquid-filled container

Liquid sloshing in moving containers is a very important problem in engineering. Fluid storage tanks during earthquakes and aircraft oil tanks subjected to dynamic loads are examples of this kind of problem. Early work relevant to this topic was performed by Moiseev[1] and Buseck, et al.[2], and others. An experimental and analytical study for the slosh response including the large wave height in horizontal cylindrical tanks was carried out by Kobayashi[3]. Gou Xingyu[4] studied forced sloshing of liquid in a narrow rectangular container, while Everstine[5] developed the analogy between the equations of elasticity and the common equations of classical mathematical physics so that it is possible for existing general FEM structural analysis programs to be used to solve a variety of field problems. However, the 6×6 material matrix relating the six stress components to the corresponding strain components is approximately constructed by Everstine and is not positive definite.

The liquids discussed here are invisible and irrotational. These liquids are also assumed to undergo only small motions. In this paper, the analogy between the wave equation of liquid and the Navier equations of structural elasticity is examined in detail. By introducing appropriate elastic constants and mass density, prescribing adequate force boundary conditions, associating additional surface mass, it is easy for us to construct the structural counterpart of the liquid sloshing model. The 6×6 material matrix derived here is not only exact but in fact a unit matrix. As a result, the dynamic analysis of liquid sloshing can be reduced to that of structural elasticity. The method presented can help us to understand the feature of liquid sloshing from the viewpoint of structural elasticity. And standard general structural analysis FEM codes can be applied, without any modification, to the problem of dynamic analysis of liquid sloshing. The numerical example shows that the results obtained with the proposed method are in good agreement with the theoretical ones.

1 The Analogy between the Governing Differential Equations

The governing differential equation for the amplitude of pressure $p(x,y,z)$ in a compressible, invisid and irrotational liquid with small dynamic motion is the following equation:

$$\nabla p = \frac{\rho_f}{B}\frac{\partial^2 p}{\partial t^2} \qquad \text{in } V \tag{1}$$

with the boundary conditions

$$\frac{\partial p}{\partial n} = -\rho_f \boldsymbol{a}\cdot\boldsymbol{n} \qquad \text{at } S_{\mathrm{w}} \tag{2}$$

$$\frac{\partial p}{\partial n} = -\frac{1}{g}\frac{\partial^2 p}{\partial t^2} \qquad \text{at } S_{\mathrm{f}} \tag{3}$$

where ∇ is the Laplacian operator; ρ_{f} is the liquid mass density; B is the liquid bulk modulus; $\boldsymbol{a}$ is the acceleration vector of the container; $\boldsymbol{n}$ is the unit outward normal vector to the liquid boundary; g is the gravitational acceleration; V is the liquid body; S_{w} is the liquid-container interface; and S_{f} is the liquid free surface.

On the other hand, if we assume that the

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stress-strain matrix for orthotropic materials is as follows

$$\boldsymbol{D} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & & & \\ D_{12} & D_{22} & D_{23} & & & \\ D_{13} & D_{23} & D_{33} & & & \\ & & & G_{xy} & & \\ & & & & G_{yz} & \\ & & & & & G_{zx} \end{bmatrix} \tag{4}$$

then the z component of the Navier equations of elasticity which are equations solved by structural analysis FEM programs is

$$G_{zx}\frac{\partial^2 w}{\partial x^2} + G_{yz}\frac{\partial^2 w}{\partial y^2} + D_{33}\frac{\partial^2 w}{\partial z^2} + (G_{zx} + D_{13})\frac{\partial^2 u}{\partial x \partial z} + (G_{yz} + D_{23})\frac{\partial^2 v}{\partial y \partial z} = \rho_s \frac{\partial^2 w}{\partial t^2} \tag{5}$$

where u, v and w are displacements in x, y and z directions, respectively; ρ_s is the mass density of the elasticity. By introducing the following conditions

$$u = v = 0 \tag{6}$$

$$G_{zx} = G_{yz} = D_{33} = 1 \tag{7}$$

$$\rho_s = \frac{\rho_f}{B} \tag{8}$$

and replacing w with p, Eq. (5) can be reduced to Eq. (1) no matter what value are D_{13} and D_{23}. Therefore, if we assume the liquid body to be a structural elasticity satisfying Eqs. (6), (7) and (8), the z component of the displacement in the elasticity is equal to the pressure in the liquid body in value. It should be noted that if the material constants for the structural elasticity are defined as

$$\nu_{xy} = \nu_{yz} = \nu_{zx} = 0 \tag{9}$$

$$E_x = E_y = G_{xy} = c \tag{10}$$

$$E_z = G_{yz} = G_{zx} = 1 \tag{11}$$

where c is any constant larger than zero (for example $c = 1$); ν, E and G are Possion's ratio, elastic constant and shear modulus of the structural elasticity, respectively. As a result, $D_{12} = D_{13} = D_{23} = 0$, $D_{11} = D_{22} = D_{33} = 1$, and $\boldsymbol{D}$ is a unit matrix. Obviously, such a $\boldsymbol{D}$ is positive and satisfies Eq. (7) exactly. In conclusion, Eqs. (6), (8), (9), (10) and (11), with w replaced by p, form the structural model corresponding to Eq. (1).

2 Structural Boundary Conditions Corresponding to Liquid Boundary Conditions

Noting Eq. (6) and that the stress-strain matrix $\boldsymbol{D}$ is unit, the z component of the force boundary conditions of the above structural elasticity is

$$n_x\frac{\partial w}{\partial x} + n_y\frac{\partial w}{\partial y} + n_z\frac{\partial w}{\partial z} = T_z \tag{12}$$

where n_x, n_y and n_z are cosine of the outward normal of the elasticity boundary; T_z is the z component of the surface force acting on the elasticity boundary. On the other hand,

$$n_x\frac{\partial w}{\partial x} + n_y\frac{\partial w}{\partial y} + n_z\frac{\partial w}{\partial z} = \frac{\partial w}{\partial n} \tag{13}$$

Thus, Eq. (12) can be written as

$$\frac{\partial w}{\partial n} = T_z \tag{14}$$

2.1 The structural force boundary condition on the liquid-container interface

If Eq. (14) is compared with Eq. (2), with w replaced by p and T_z replaced by $-\rho_f \boldsymbol{a} \cdot \boldsymbol{n}$, Eq. (14) reduces to Eq. (2). Hence, the structural force boundary condition corresponding to Eq. (2) is

$$T_z = -\rho_f \boldsymbol{a} \cdot \boldsymbol{n} \qquad \text{at } S_w \tag{15}$$

2.2 The structural boundary condition on the free surface of liquid

In a similar way, if Eq. (14) is compared with Eq. (3), with w replaced by p and T_z replaced by $-\frac{1}{g}\frac{\partial^2 w}{\partial t^2}$, Eq. (14) becomes Eq. (3). Therefore, the structural force boundary condition corresponding to Eq. (3) is

$$T_z = -\frac{1}{g}\frac{\partial^2 w}{\partial t^2} \qquad \text{at } S_f \tag{16}$$

It should be noted that in present practice, $\frac{\partial^2 w}{\partial t^2}$ is the z component of the free surface acceleration. In order to exert a surface force expressed by Eq. (16), the only thing we must do is to associate an additional surface mass of $\frac{1}{g}$ with surface S_f.

3 Numerical Example

In order to illustrate the effectiveness of the present method, the sloshing frequencies and modes of a cylindrical tank filled with liquid are calculated as an example. The main parameters are as follows: the radius of the cylindrical container $R = 0.5$ m, the height of filled liquid (water) $H = 0.6$ m, the density of water $\rho_f = 1\,000$ kg/m^3. If water is assumed incompressible, some of the sloshing modals are exhibited in Fig. 1 (m is the number of radial half wave and n is the number of circular wave). Tab. 1 shows some of the obtained sloshing frequencies ($n = 1$), which are in excellent agreement with the analytical solution under incompressible liquid condition given by the following expression[6]

$$f_m = \frac{1}{2\pi}\sqrt{\frac{g\sigma_m}{R}\tanh(\frac{H\sigma_m}{R})} \qquad m = 1,2,3,\cdots \quad (17)$$

where $\sigma_m (m = 1,2,3,\cdots)$ are the roots of $J'_1(\sigma_m) = 0$ while J_1 is the Bessel function of the first order.

On the other hand, if the bulk modulus of water is taken as $B = 2.0$ GPa, the results obtained are almost the same as above. This shows that the compressibility of liquid affects the sloshing frequencies and modes little.

Tab.1 Comparison of sloshing frequencies

$m(n = 1)$	Method in this paper	Analytical solution	Computed error/%
1	0.945 6	0.944 3	0.14
2	1.628 9	1.626 7	0.14
3	2.064 2	2.057 9	0.31
4	2.423 0	2.411 2	0.49

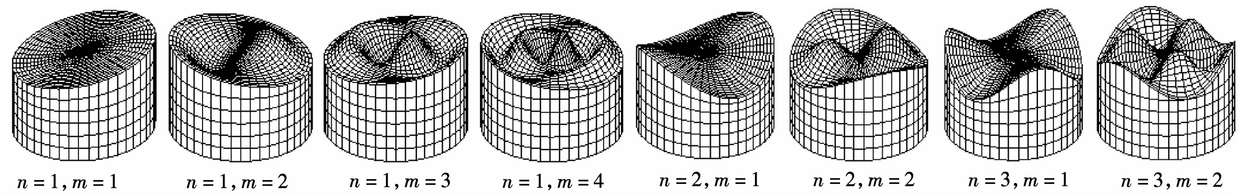

Fig.1 Liquid sloshing modals in the cylinder

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结构类比法进行液体晃动动力学分析

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摘　要　深入分析了液体波动方程与弹性体 Navier 方程及其边界条件之间的对应相似关系,运用比拟法构造了对应于液体晃动的结构弹性体模型,将液体晃动动力问题归结为结构弹性体动力问题.算例结果与解析解的比较表明,文中所述方法十分可靠.

关键词　液体晃动, 动力分析, 充液容器

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