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Robust Delaunay Tetrahedronal Meshing Coupled with Advancing Front Method

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Abstract: A full automatic tetrahedronal mesh generation method for arbitrary 3D domains is described. First, the classic Delaunay method is coupled with simplified advancing front technique (AFT) to obtain the boundary mesh. Then, advancing-front high-quality point placement is used to generate internal points with optimal positions and a Delaunay method is used to insert them efficiently. Finally, optimization procedures are used for mesh quality improvements. Several application examples are presented to demonstrate the robustness and efficiency of the proposed meshing scheme.

Key words: tetrahedronal meshing, Delaunay method, advancing front technique, boundary meshing

Since the works of C. Lawson and D. Watson, Delaunay triangulation has been investigated for many years. It has been developed to be very mature in 2D meshing. But for 3D cases, there still have remained one unresolved bottleneck: boundary integrity or boundary recovery. Many researchers have proposed several treatments for it: adding extra points (usu. mid-edge points) i.e. refinement; local edges/shell operation (like swapping)[1,2]. However, among the above methods, local operation makes use of heuristics; refinement treatment leads to surfaces triangulation incompatibility when merging several singly meshed bodies. In this paper, we couple the Delaunay method with a simplified advancing front method for obtaining the boundary tetrahedronization which observes the boundary integrity. Also, based on this initial tetrahedronization, we again couple the advancing front points generation with the Delaunay points insertion to complete the required tetrahedronization robustly and efficiently.

Before the boundary tetrahedronization, we perform a Delaunay tetrahedronization of the convex hull of the boundary points and specified points, from which a control space(or sizing spacing) and a semi-boundary tetrahedronization are derived. Then the classic advancing front method is simplified: the candidate points for each frontal face are only the surface points; the optimal choice of the candidate point depends solely on the element quality of the so-formed element; the validity testing is also reduced to the faces' intersection checking[3-6]. When we encounter the cases that no point allows to create a valid element for a given frontal face, we advance from another frontal face. If this still cannot work, we delete the tetrahedrons connected to the problematic face if it is an internal face or we create a Steinor point with optimal position(with respect to the frontal face)by making use of the control space if it is a boundary face. In this way, comparing with the whole meshing process, we can inexpensively obtain the boundary tetrahedronization.

Once the boundary tetrahedronization is completed, we, front by front, iteratively use the advancing front technique to create the inner points with optimal positions(also making use of the control space) and then insert them into the existing tetrahedronization with the optimal Delaunay insertion procedures[1,7-9]. This couplement combines the advantages of efficiency and nice mathematical properties of a Delaunay method and AFT front high-quality point placement[7]. At last, we use optimized Lapacian smoothing and local transformation mesh improvements.

In the following sections, section 1 will discuss the first couplement of the Delaunay method and AFT

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for obtaining the boundary mesh. Section 2 will deal with the second couplement for interior refinement(i.e. points generation and insertion). Section 3 will present the mesh optimization procedures. Section 4 will summarize the meshing scheme and provide two meshing examples: a complex geometric model and a large transformer. The robustness and efficiency of the generation procedure and the high quality of the elements (tetrahedrons) illustrate that this coupling is a nice mesh generation solution.

1 Boundary Tetrahedronization

Our boundary tetrahedronization consists of two steps: first, we use the Delaunay method to form a semi-boundary tetrahedronization; then, we apply AFT to complete the boundary meshing.

1.1 Semi-boundary tetrahedronization

The data necessary for our mesh generator is the surface triangulation (of the boundary of the domain) whose elements are oriented with the normals pointing to the interior of the domain. Let SP be the set of the nodes of the surface triangulation. For the semi-boundary tetra-hedronization (SBT), firstly, we perform a standard Delaunay tetrahedronization T^* of SP and this mesh forms a convex hull of SP; secondly, we divide the tetrahedra ST of T^* into two parts: BST and IST, among which BST are the tetrahedra (of ST) each of which has at least one boundary face(in respect to T^*, and we call the face its skin face), and IST the inner tetrahedra. For each element of BST, if each of its skin faces coincides with one triangle of the surface triangulation and this tetrahedron doesn't intersect any element of the surface triangulation, we mark this tetrahedron valid, else invalid. For each element of IST, only if it passes the above intersection checks, we mark it valid, else invalid; thirdly, we delete the invalid elements and retain the valid elements from T^*; finally, we add the surface triangulation to the valid elements and obtain the semi-boundary tetrahedronization.

Obviously, SBT has "empty areas" due to the deletion of the invalid elements. Usually, the volume of these "areas" is only a small percentage of that of the whole domain. Hence, we can use AFT to "fill" these "empty areas" with a comparatively small number of tetrahedra, i.e., inexpensively to complete the boundary tetrahedronization.

1.2 Complete boundary tetrahedronization

In our simplified AFT, the shape quality criterion for a tetrahedron is defined as $Q(t) = \alpha(\rho/h)$ where h is the longest edge length of the element t; ρ is the radius of the inscribed sphere of t; a coefficient α is applied so that the highest criterion(of equilateral element) is 1.

The process of the simplified AFT goes as follows:

① Initialize the front to be the addition of the boundary faces(of SBT) each of which connects no tetrahedron and the inner faces(of SBT) each of which connects only one tetrahedron;

② Select an element from the front as a base face;

③ Find the candidate points set A: A consists of the frontal points which lie in the left side of the base face, i.e., the tetrahedron the base face and the candidate point form has a positive volume. And with A, we form the candidate tetrahedra set $T(A)$;

④ Sort $T(A)$ in the order of descending element shape quality $Q(t)$(t of $T(A)$);

⑤ Beginning from the first element of $T(A)$, try to find the first element e^*, which doesn't intersect any element of the front;

⑥ If e^* exists, its corresponding point P^* (in A) is set to be the candidate point we want and go to ⑧; else, we choose a different element of the front as the base face and go to ②;

⑦ If all the elements of the front have been tried and failed, we encounter the Schronder case. For this, we have the following strategy[4].

a. If the current base face is an inner face, delete the tetrahedra connecting it and update the front go to ②;

b. Else add a Steiner point Q on the vertical to the centre of gravity of the base face and delete all the intersection (if it occurs) tetrahedra. Let $P^* = Q$, go to ⑧.

⑧ Form a tetrahedron with P^* and the base face and add it to the data structure;

⑨ Update the front. If the front is not empty, go to ③.

After the boundary tetrahedronization, we derive a control space (or sizing space) through interpolation and construct a neighborhood grid. Also we define a normal edge length for an edge in the method introduced by George[7].

2 Interior Refinement

Once we have obtained the boundary mesh, the successive work is to generate interior points and insert them into the existing mesh. We use the method proposed by P.L.George: generate nodes with AFT and insert them in Delaunay-based methods. Here, we just recall the outline of the algorithm.

① Initialize the front F_i ($i = 0$) to be the boundary faces of the boundary mesh, and mark all tetrahedra to be unacceptable (i.e. the shape quality measure is below a priorior criterion, else, acceptable). Let A_i = (all the acceptable elements), U_i = (all the unacceptable elements). Obviously $A_0 = T_0$(T_i is the tetrahedra of the ith stage).

② Find an optimal point for each element e^* of F_i. This point lies on the same side of the face e^* as the unacceptable element leaning on it, and is placed at a position which is determined to form an optimal tetrhedron with e^*. Using the control space and the notion of normalize length, we carry out an iterative procedure to construct the optimal point. These points form the points cloud $N_i + 1$.

③ Filtering of the $N_i + 1$. Remove from $N_i + 1$ a point which violates the size criterion (via the control space and neighboring grid space) when compared with a previous selected point form $N_i + 1$.

④ If the filtered point cloud $N_i + 1$ is empty, go to end.

⑤ Insertion of the retained points into T_i via the constrained Delaunay insertion procedure[1,2,8].

⑥ Update the front F_i and go to ⑨. Once the T_i is constructed, we classify the elements of T_i into A_i and U_i. Then, $F_i = (f{:}f = (k_1, k_2)$, where k_1 is in A_i and k_2 is in U_i), i.e. f is a face between two elements, one is acceptable, the other is unacceptable.

3 Mesh Improvement

Good mesh quality is a major key to obtain precise solution (or to facilitate the solution step of the computation, in particular, when an iterative method is used). In our mesh generation, even though the AFT optimal field points generation has governed the entire interior refinement, in 3D, this is not sufficient for satisfactory mesh quality. Hence, mesh quality improvement is a necessary stage. Here we briefly describe two kinds of optimization procedures: vertex relocating and local mesh modifications.

3.1 Vertex relocating

Laplacian smoothing and optimal smoothing are usually applied for vertex shifting. We use optimal smoothing. This method was introduced by George[2]. It consists in moving the mesh vertexes to increase the element quality. And it is applied to the free vertexes which do not belong to the boundary surfaces or any constrained inner surface.

Let P be a free vertex, and $B(P)$ be the ball associated with P(i.e. the set of elements sharing P). $B(P)$ constitutes a star-shaped polyhedron with respect to P. Let F be the boundary faces of the polyhedron surrounding the elements of $B(P)$. For each face f of F, we define an "optimal" point P_f, so that the tetrahedron based on f is optimally shaped and the latter is in the same side of f with respect to P. Then we move P "step-by-step" to the centroid of $B(P)$, if the quality of the worst element is improved.

3.2 Local mesh modification

This optimization stage consists of improving the element quality using local topological mesh modifications. We use the methods introduced by Rassineux[4].

① Meshing around an edge. Selecting an edge of a bad-quality tetrahedron and remeshing the shell consisting of all tetrahedrons which share this edge.

② Meshing a shell by deletion. Selecting a face(not constrained or boundary) of a bad-quality tetrahedron and remeshing the shell which consists of all the tetrahedrons sharing this face into three tetrahedra.

③ Meshing by nodal insertion. Improving the quality of the shell consitited by all the tetrahedra sharing the edge by remeshing around free edges (not boundary or constrained), which generating a node at the center of the shell.

④ Meshing by nodal deletion. Coarsening the mesh by remeshing the shell constituted by all the tetrahedra sharing the same node and delete the node.

In our global optimization procedure, we first apply the optimal Lapacian smoothing for vertex relocating; then, we search for the bad-quality tetrahedra and perform one or several (in proper order) of the above local transformation procedures, and finally apply the optimal Lapacian smoothing again. We find this combination of optimization procedures is very effective.

4 Meshing Scheme

As a summary of the previous sections, we have the following mesh scheme.

Firstly, we create the surface triangulation of the boundary data and specified items of the domain and make a proper treatment of the triangulation. Then

Step 1 Apply the Delaunay method to the boundary points and specified field points to form a tetrahedronzation of the convex hull of these points.

Step 2 Form a semi-boundary mesh of the formed convex tetrahedronization through retaining the valid element, deleting the invalid elements and adding the surface triangulation.

Step 3 Apply a simplified version of the advancing front method to complete the boundary meshing.

Step 4 Use the advancing front method to generate the inner points and apply the Delaunay method to insert them into the boundary mesh.

Step 5 Apply a combination of optimal Lapacian smoothing and local mesh transformation to improve the mesh quality.

5 Application Examples

We have selected two examples of meshes created using the proposed method to demonstrate the robustness and efficiency of it. The first example is a meshing of a complex geometric model with several very narrow air gaps, of which the mesh has large element size-ratio. The second one is a half of a large transformer, of which the mesh has local refinement in the iron core. Fig. 1 and Fig. 2 show the two meshes respectively.

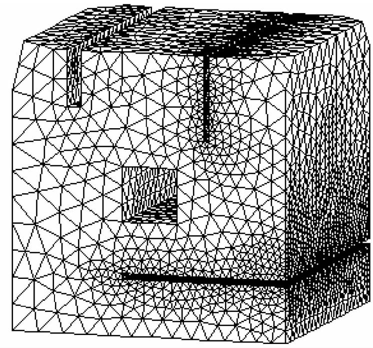

Fig. 1 A tetrahedronal mesh of a complex geometric model with very narrow air gaps

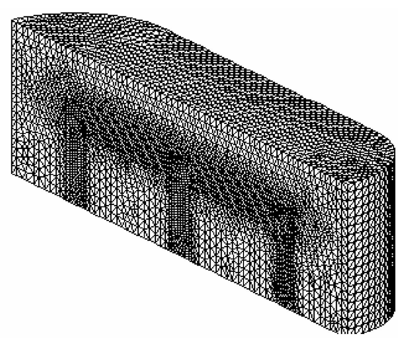

Fig. 2 A tetrahedronal mesh of a half of a large transformer with mesh refinement in the iron core

6 Concluding Remark and Future Work

We have presented a Delaunay-based method which is coupled with advancing front method for meshing arbitrary 3D domains. The first couplement of Delaunay meshing and AFT tetrahedronization solves the hardnut: boundary integrity and get the boundary mesh. The second couplement combines the advantages of efficiency and nice mathematical properties of a Delaunay approach and the advancing front high-quality point placement strategy. The optimization procedures improve the mesh quality. Complex numerical examples have been shown to illustrate the capability and robustness of the meshing scheme.

Future work, in short, is as follows.

1) The improvement of the element deletion and Steiner points addition strategy for Schrondert cases.

2) The improvement of the optimization

procedure.

3) Extension to the isotropic and ansitropic cases.

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与波前法相结合的 Delaunay 四面体网格生成方法

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摘　要　描述了一种在任意三维区域自动生成四面体网格的方法.首先,将经典的 Delaunay 方法与简化的波前法相结合,进行边界四面体剖分,解决边界还原问题.然后,再次将 Delaunay 方法与波前法相结合,产生具有最优位置的内部节点,并用 Delauny 方法将内部节点高效率地插入.最后,进行网格优化以提高网格的质量.文中的应用算例显示,本文提出的网格生成方法具有很强的健壮性和高效率.

关键词　四面体网格, Delaunay 方法, 波前法, 边界网格生成

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