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Quasi-χ^2 Distribution and the Independence of Wishart Distribution

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Abstract: In this paper, the authors generalize the definition of χ^2 distribution and introduce a quasi-χ^2 distribution, and then prove several properties of it, find the necessary and sufficient conditions of independence about multivariate normal distributions, matrix normal distributions and two parts of the Wishart distribution.

Key words: quasi-χ^2, Wishart distribution, independence

Definition Let $x \sim N_n(\mu, \Lambda_n)$, $\Lambda_n = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$, $\lambda_j \geqslant 0$, $j = 1, \cdots, n$, then $x'x$ is a quasi-χ^2 distribution. And we denote it as $x'x \sim g\chi^2(\mu, \Lambda_n)$. If $\mu = 0$, then it is written as $x'x \sim g\chi^2(\Lambda_n)$. Obviously, if $\Lambda_n = I_n$, then quasi-χ^2 distribution is a χ^2 one. It's obtained that quasi-χ^2 distribution generalizes the χ^2 one. First an important lemma proved in Ref.[4] is introduced.

Lemma 1 $\dfrac{1}{\sqrt{2\pi\lambda}}\displaystyle\int_{-\infty}^{+\infty} \mathrm{e}^{itx^2 - \frac{(x-\mu)^2}{2\lambda}}\mathrm{d}x = (1-2i\lambda t)^{-\frac{1}{2}} \times \exp\left\{\dfrac{i\mu^2 t}{1-2i\lambda t}\right\}$, $\lambda > 0$.

Then the following properties of quasi-χ^2 distribution are obtained.

Proposition 1 If $z \sim g\chi^2(\mu, \Lambda_n)$, $\mu = (\mu_1, \cdots, \mu_n)'$, $\Lambda_n = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$, then the characteristic function (c.f.) of z is $\Phi_z(t) = \exp\left\{\displaystyle\sum_{j=1}^{n} \dfrac{i\mu_j^2 t}{1-2i\lambda_j t}\right\}\displaystyle\prod_{j=1}^{n}(1-2i\lambda_j t)^{-\frac{1}{2}}$. The proof can be found in page 141 of Ref.[4].

Corollary 1 If $z \sim g\chi^2(\Lambda_n)$, then the c.f. of z is $\Phi_z(t) = \displaystyle\prod_{j=1}^{n}(1-2i\lambda_j t)^{-\frac{1}{2}}$.

Corollary 2 If $z \sim g\chi^2(\mu, \Lambda_n)$, $z + a \sim g\chi^2\left(\begin{pmatrix}\mu \\ \mu_{n+1}\end{pmatrix}, \begin{pmatrix}\Lambda_n & 0 \\ 0 & 0\end{pmatrix}\right)$, where a is a positive constant with $\mu_{n+1}^2 = a$. From this, the condition with $\lambda_{n+1} = 0$ can be changed into the addition of a constant and a $g\chi^2$ variable with all λ_j are positive. So in the following discuss, we let $\lambda_j > 0$. $j = 1, \cdots, n$.

Proposition 2 If $z \sim g\chi^2(\mu, \Lambda_n)$, $\mathrm{E}z = \mu'\mu + \mathrm{tr}(\Lambda_n)$, $\mathrm{Var}(z) = 4\mu'\Lambda_n\mu + 2\mathrm{tr}(\Lambda_n^2)$.

Proof We get it through the differentiation on of the c.f. of z.

Proposition 3 Let $x \sim N_n(\mu, \Sigma)$, then $x'x \sim g\chi^2(P'\mu, \Lambda_n)$, where $\Lambda_n = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$, and λ_i, $i = 1, \cdots, n$ are the latent roots of Σ. P is orthogonal matrix with the standardized latent vectors corresponding to Σ.

Proof For $\Sigma \geqslant 0$, we make a transformation from Σ to its latent roots and vectors i.e., put $P'\Sigma P = \Lambda_n \geqslant 0$, where $P \in O(n)$ and $\Lambda_n = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$. Let $y = P'x \sim N_n(P'\mu, \Lambda_n)$, by the definition, we get $x'x = y'y \sim g\chi^2(P'\mu, \Lambda_n)$.

Lemma 2 If $P'\begin{pmatrix}\Lambda_{n_1} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{n_2}\end{pmatrix}P = \begin{pmatrix}\Lambda_{n_1} & 0 \\ 0 & \Lambda_{n_2}\end{pmatrix}$, where Λ_{n_α} is the positive defined diagonal matrix and $P \in O(n)$, $\Lambda_{12} = \Lambda'_{21}$, then $\Lambda_{12} = 0$.

Proof By page 67 in Ref.[4], the square sum of matrix's elements does not change under orthogonal transformation. So the square sum of the elements in Λ_{12} is zero, this holds $\Lambda_{12} = 0$.

Proposition 4 let $z_1 \sim g\chi^2(\mu_1, \Lambda_{n_1})$, $z_2 \sim g\chi^2(\mu_2, \Lambda_{n_2})$, $z_1 = x'_1 x_1$, $z_2 = x'_2 x_2$, $x_i \sim N(\mu_i, \Lambda_{n_i})$, $i = 1, 2$, then z_1 is independent of z_2 if and only if $z_1 + z_2 \sim g\chi^2(\mu, \Lambda)$, and the joint distribution of x_i is a normal distribution, $\mu = (\mu'_1, \mu'_2)'$, and $\Lambda = \mathrm{diag}(\Lambda_{n_1}, \Lambda_{n_2})$.

Proof ① **The sufficiency** Let $z_1 + z_2 \sim g\chi^2(\mu, \Lambda)$ and $\mu_\alpha = (\mu_{\alpha 1}, \cdots, \mu_{\alpha n_\alpha})'$, $\Lambda_{n_\alpha} = \mathrm{diag}$

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$(\lambda_{\alpha1},\cdots,\lambda_{\alpha n_\alpha})$, $\alpha = 1,2$, then from the definition of $g\chi^2$ and the above conditions, there exist two random vectors x_1, x_2 which satisfy $z_1 = x_1' x_1$, $z_2 = x_2' x_2$. $x = (x'_1, x'_2)' \sim N_{n_1+n_2}(\mu, \Lambda_n)$ where $\Lambda_n = \begin{pmatrix} \Lambda_{n_1} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{n_2} \end{pmatrix}$.

Note that $z = z_1 + z_2 = x'x \sim g\chi^2(\mu, \Lambda)$, but by proposition 3, $z = x'x \sim g\chi^2(\mu^*, \bar{\Lambda}_n)$, where $\mu^* = P'\mu$, $P = (P_1, P_2)$ is an orthogonal matrix, and p_i is the latent vector of Λ_n, $\bar{\Lambda}_n = \operatorname{diag}(\lambda_1, \cdots, \lambda_{n_1+n_2})$, and λ_i is the latent root for p_i.

For the c.f. of $g\chi^2$ is unique, by comparing the singular points of the c.f. The latent roots must be λ_{α_j}, $\alpha = 1,2, j = 1,\cdots,n_\alpha$, they follows that $\begin{pmatrix} P'_1 \\ P'_2 \end{pmatrix} \Lambda_n (P_1, P_2) = \bar{\Lambda}_n = \Lambda$. From the lemma 2, $\Lambda_{12} = 0$ is obtained.

So x_1 is independent of x_2, i.e. z_1 and z_2 are independent.

② **The necessity** If z_1 is independent of z_2, we get the equation

$$\Phi_{z_1+z_2}(t) = \Phi_{z_1}(t)\Phi_{z_2}(t) = \prod_{\alpha=1}^{2} \prod_{j=1}^{n_\alpha} (1 - 2i\lambda_{\alpha j} t)^{-\frac{1}{2}} \exp\left\{ \sum_{\alpha=1}^{2} \sum_{j=1}^{n_\alpha} \frac{i\mu_{\alpha j}^2 t}{1 - 2i\lambda_{\alpha j} t} \right\},$$ so $z_1 + z_2 \sim g\chi^2(\mu, \Lambda)$.

Theorem 1 Let $x_1 \sim N_{n_1}(\mu_1, \Sigma_1)$, $x_2 \sim N_{n_2}(\mu_2, \Sigma_2)$, then x_1 is independent of x_2 if and only if $x'_1 x_1$ is independent of $x'_2 x_2$ and the joint distribution of x_1 and x_2 is a multivariate normal distribution.

① **The necessity** If x_1 is independent of x_2, then $x_1' x_1$ is independent of $x_2' x_2$, and the joint distribution of x_1 and x_2 is the product of their density functions, so the joint distribution of x_1 and x_2 is a multivariate normal distribution.

② **The sufficiency** For $\Sigma_\alpha \geqslant 0$, there exist an orthogonal matrix P_α such that $P'_\alpha \Sigma_\alpha P_\alpha = \Lambda_{n_\alpha}$, $\Lambda_{n_\alpha} = \operatorname{diag}(\lambda_{\alpha1}, \cdots \lambda_{\alpha n_\alpha})$, $\alpha = 1,2$. Let $y_\alpha = P'_\alpha x_\alpha$, $\alpha = 1,2$, Since x_1, x_2 have normal distribution

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} P'_1 & 0 \\ 0 & P'_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \sim N_{n_1+n_2}\left(\begin{pmatrix} P'_1\mu_1 \\ P'_2\mu_2 \end{pmatrix}, \begin{pmatrix} \Lambda_{n_1} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{n_2} \end{pmatrix} \right)$$

So $y'_\alpha y_\alpha \sim g\chi^2(P'_\alpha \mu_\alpha, \Lambda_{n_\alpha})$, $\alpha = 1,2$. Note that $x_1' x_1$, $x_2' x_2$ are independent, and $x'_\alpha x_\alpha = y'_\alpha y_\alpha$, $\alpha = 1,2$. Therefore $y_1' y_1$ and $y_2' y_2$ are independent, then from the proof of the necessity of proposition 4, $y_1' y_1 + y'_2 y_2$ is a $g\chi^2\left(\begin{pmatrix} P'_1\mu_1 \\ P'_2\mu_2 \end{pmatrix}, \begin{pmatrix} \Lambda_{n_1} & 0 \\ 0 & \Lambda_{n_2} \end{pmatrix} \right)$.

From lemma 2 $\Lambda_{12} = 0$, so y_1 and y_2 are independent, i.e. x_1 and x_2 are also independent.

Corollary 3 Let $x \sim N_n(\mu_1, \Sigma_1)$, $y \sim N_m(\mu_2, \Sigma_2)$, then x and y are independent if and only if $x'x$ and $y'y$ are independent and the joint distribution of x and y is a multivariate normal distribution.

Corollary 4 Let $x \sim N_n(\mu_1, \Sigma_1)$, $y \sim N_m(\mu_2, \Sigma_2)$, $x'x \sim \chi_p^2(\delta_1)$, $y'y \sim \chi_q^2(\delta_2)$, then x is independent of y if and only if $x'x + y'y \sim \chi_{p+q}^2(\delta_1 + \delta_2)$, and the joint distribution of x and y is a multivariate normal distribution.

Proof ① **The sufficiency** If $x'x + y'y \sim \chi^2_{p+q}(\delta_1 + \delta_2)$, $x'x$ is independent of $y'y$ by Cochran theorem, hence x and y are independent due to theorem 1.

② **The Necessity** If x is independent of y, the joint distribution of x and y is a multivariate normal distribution, and $x'x$ is independent of $y'y$, $x'x \sim \chi_p^2(\delta_1)$, $y'y \sim \chi_q^2(\delta_2)$, therefore $x'x + y'y \sim \chi^2_{p+q}(\delta_1 + \delta_2)$.

Theorem 2 Let $X \sim N_{n\times p}(M, W \otimes V)$, $Y \sim N_{m\times q}(D, A \otimes B)$, then X is independent of Y if and only if $X'X$ is independent of $Y'Y$ and the joint distribution of $\operatorname{Vec}X$ and $\operatorname{Vec}Y$ is a multivariate normal distribution.

Proof ① **The necessity** If X and Y are independent, then $X'X$ and $Y'Y$ are independent, and $\operatorname{Vec}X$ is independent of $\operatorname{Vec}Y$, Since $\operatorname{Vec}X \sim N_{np}(\operatorname{Vec}M, V \otimes W)$, $\operatorname{Vec}Y \sim N_{mq}(\operatorname{Vec}D, B \otimes A)$, the joint distribution of $\operatorname{Vec}X$ and $\operatorname{Vec}Y$ is a multivariate normal distribution.

② **The sufficiency** If $X'X$ is independent of $Y'Y$, then $\operatorname{tr}(X'X)$ is independent of $\operatorname{tr}(Y'Y)$, i.e., $\sum_i \sum_j x_{ij}^2 = (\operatorname{Vec}X)'\operatorname{Vec}X$ and $\sum_s \sum_t y_{st}^2 = (\operatorname{Vec}Y)'\operatorname{Vec}Y$ are independent. The joint distribution of $\operatorname{Vec}X$ and $\operatorname{Vec}Y$ is a multivariate normal distribution. From theorem 1, $\operatorname{Vec}X$ and $\operatorname{Vec}Y$ are independent, hence X is independent of Y.

Corollary 5 Let X and Y satisfy the conditions of theorem 2, then X is independent of Y if and only if $\operatorname{tr}(X'X)$ is independent of $\operatorname{tr}(Y'Y)$, and the joint distribution of $\operatorname{Vec}X$ and $\operatorname{Vec}Y$ is a multivariate normal.

Theorem 3 Let $W \sim W_P(n, \Sigma, \Delta)$, $W =$

$\begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}$, $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$, where W_{11} and Σ_{11} are $q \times q$ matrixes, W_{11} and W_{22} are Wishart distributions then W_{11} is independent of W_{22} if and only if $\Sigma_{12} = 0$.

Proof From the definition of $W_P(n, \Sigma, \Delta)$, there exists $x \sim N_{n\times p}(M, I \otimes \Sigma)$ such that $W = x'x$, where $M'M = \Delta$. Partitioning X and M as $x = (\underset{q}{x_1}, \underset{p-q}{x_2})$, $M = (\underset{q}{M_1}, \underset{p-q}{M_2})$, so $x_1 = (x_1\ x_2)\begin{pmatrix} I_q \\ 0 \end{pmatrix} \sim N_{n\times q}\left((M_1\ M_2)\begin{pmatrix} I_q \\ 0 \end{pmatrix}, I \otimes \left((I_q \quad 0)\Sigma\begin{pmatrix} I_q \\ 0 \end{pmatrix}\right)\right) = N_{n\times q}(M_1 \quad I \otimes \Sigma_{11})$ and $x_2 \sim N_{n\times(p-q)}(M_2, I \otimes \Sigma_{22})$. This shows that

$$W_{11} = (I_q \quad 0)\begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}\begin{pmatrix} I_q \\ 0 \end{pmatrix} = (I_q \quad 0)\begin{pmatrix} x_1' \\ x_2' \end{pmatrix}(x_1 \quad x_2)\begin{pmatrix} I_q \\ 0 \end{pmatrix} = x_1' x_1$$

and $W_{22} = x_2' x_2$. On the other hand, $\mathrm{Vec}x \sim N_{np}(\mathrm{Vec}M, \Sigma \otimes I)$ leads to $\begin{pmatrix} \mathrm{Vec}x_1 \\ \mathrm{Vec}x_2 \end{pmatrix} \sim N_{np}\left(\begin{pmatrix} \mathrm{Vec}M_1 \\ \mathrm{Vec}M_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} \otimes I & \Sigma_{12} \otimes I \\ \Sigma_{21} \otimes I & \Sigma_{22} \otimes I \end{pmatrix}\right)$.

If $\Sigma_{12} = 0$, we have $\Sigma_{12} \otimes I = 0$, then $\mathrm{Vec}x_1$ is independent of $\mathrm{Vec}x_2$, hence x_1 is independent of x_2 so $x'_1 x_1$ is independent of $x'_2 x_2$. i.e., W_{11} is independent of W_{22}.

The sufficiency If W_{11} is independent of W_{22}, from the above discussion, $x_1' x_1$ is independent of $x_2' x_2$, and $x = (x_1, x_2)$ is normal random matrix. Therefore $\mathrm{Vec}x = \begin{pmatrix} \mathrm{Vec}x_1 \\ \mathrm{Vec}x_2 \end{pmatrix}$ is a multivariate normal distribution, $\mathrm{Vec}x_1$ and $\mathrm{Vec}x_2$ are independent by theorem 2. This leads $\Sigma_{12} \otimes I = 0$, hence $\Sigma_{12} = 0$.

Right now we'll cite an example to demonstrate that if the joint distribution of x_1 and x_2 is not multivariate normal distribution, x_1 and x_2 cannot be independent even if $x_1' x_1$ and $x_2' x_2$ are independent.

Example Let the density function of $x = (x_1, \cdots, x_n)'$ be

$$f(x) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left(-\frac{1}{2}\sum_{i=1}^n x_i^2\right) \times \left(\mathrm{sign}\left(\prod_{i=1}^n x_i\right) + 1\right) \qquad x \in R^n$$

then the following conclusions hold.

1) $\forall m, 1 \leqslant m < n, (x_{i_1}, \cdots, x_{i_m})' \sim N_m(0, I)$, i.e., that any marginal distribution of x is standard multivariate normal distribution.

2) If $\{1, \cdots, n\} = \{i_1, \cdots, i_m\} \cup \{j_1, \cdots, j_{n-m}\}$, then $(x_{i_1}, \cdots, x_{i_m})'$ and $(x_{j_1}, \cdots, x_{j_{n-m}})'$ are not independent, so $(x_1, \cdots, x_m)'$ and $(x_{m+1}, \cdots, x_n)'$ are not independent.

3) Any m random variables of $x_1, \cdots, x_n$ $(2 \leqslant m < n)$ are independent of each other, and $x_1^2, \cdots, x_n^2$ are independent of each other, so $\sum_{k=1}^m x_{i_k}^2$ and $\sum_{k=1}^{n-m} x_{j_k}^2$ are independent.

Proof ① First prove $(x_1, \cdots x_{n-1})' \sim N_{n-1}(0, I)$. For any given $x_1, \cdots, x_{n-1}$, $\left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left(-\frac{1}{2}\sum_{i=1}^n x_i^2\right)\mathrm{sign}\left(\prod_{i=1}^n x_i\right)$, is an odd function in $(-\infty, +\infty)$ of x_n, its integral is 0. So

$$f_n(x_1, \cdots, x_{n-1}) = \int_{-\infty}^{+\infty} f(x)\mathrm{d}x_n = \int_{-\infty}^{+\infty}\left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left(-\frac{1}{2}\sum_{i=1}^n x_i^2\right)\left(\mathrm{sign}\left(\prod_{i=1}^n x_i\right) + 1\right)\mathrm{d}x_n = \int_{-\infty}^{+\infty}\left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left(-\frac{1}{2}\sum_{i=1}^n x_i^2\right)\mathrm{d}x_n = \left(\frac{1}{\sqrt{2\pi}}\right)^{n-1}\exp\left(-\frac{1}{2}\sum_{j=1}^{n-1} x_j^2\right)$$

this leads $(x_1, \cdots x_{n-1})' \sim N_{n-1}(0, I_{n-1})$.

Note that $f(x)$ is symmetric about its variables, so its any marginal vector with $n-1$ dimension is $N_{n-1}(0, I)$. And for $\forall m, 1 \leqslant m < n-1, (x_{i_1}, \cdots, x_{i_m})$ must be a portioned vector of a $(n-1)$-dimensional vector of x, this shows that $(x_{i_1}, \cdots, x_{i_m})' \sim N_m(0, I_m)$.

② From ①, the density function of $(x_{i_1}, \cdots x_{i_m})'$ and $(x_{j_1}, \cdots x_{j_{n-m}})'$ are

$$g(x_{i_1}, \cdots, x_{i_m}) = \left(\frac{1}{\sqrt{2\pi}}\right)^m \exp\left(-\frac{1}{2}\sum_{k=1}^m x_{i_k}^2\right),$$

$$h(x_{j_1}, \cdots, x_{j_{n-m}}) = \left(\frac{1}{\sqrt{2\pi}}\right)^{n-m} \exp\left(-\frac{1}{2}\sum_{k=1}^{n-m} x_{j_k}^2\right),$$

respectively.

It is easily to check that $f(x) \neq g(x_{i_1}, \cdots, x_{i_m})h(x_{j_1}, \cdots, x_{j_{n-m}})$, so $(x_{i_1}, \cdots, x_{i_m})'$ and $(x_{j_1}, \cdots, x_{j_{n-m}})'$ are not independent.

③ From ①, $(x_1, \cdots, x_{n-1})' \sim N_{n-1}(0, I_{n-1})$, so $x_1, \cdots, x_{n-1}$ are independent. And from the other $(n-1)$-dimensional marginal distributions, it's easily obtained that x_n is independent of any of $\{x_1, \cdots, x_{n-1}\}$. So any two variables of $\{x_1, \cdots, x_n\}$ are independent. But from ②, $x_1, \cdots, x_n$ are not independent of each other. Now, we will prove $x_1^2, \cdots, x_n^2$ are independent of each other.

Since, $x_i \sim N(0,1)$, $i = 1, \cdots, n$, let $y_i = x_i^2$,

$i=1,\cdots,n$, then $y_i \sim \chi_1^2$, its density function is $\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y_i}{2}\right)y_i^{-\frac{1}{2}}$, $(y_i > 0)$. From the joint density function of x, we get the joint density function of $(y_1,\cdots,y_n)$ as $(2\pi)^{-\frac{n}{2}}\exp\left(-\frac{1}{2}\sum_{i=1}^{n}y_i\right)\cdot 2^{-n}\cdot\prod_{i=1}^{n}y_i^{-\frac{1}{2}}\cdot 2^n$, $\left(|J|=\prod_{j=1}^{n}(2x_j)^{-1}\cdot 2^n\right)$ which is the product of the marginal density function, so $x_1^2,\cdots,x_n^2$ are all independent.

Actually the distribution of x is generated from an $N_n(0,I_n)$ through centralizing its density function into 2^{n-1} convex sets which are generated by the coordinate planes. Hence there are two convex sets possessed a commom coordinate plane, the density is 0 in one, and the other is the twice of the original.

References

[1] Anderson T W. *An introduction to multivariate statistical analysis*. 2nd edition[M]. New York: Wiley. 1984.

[2] Zhang Yaoting, Fang Kaitai. *An introduction to multivariate statistical analysis*[M]. Beijing: Science Publication, 1982. (in Chinese)

[3] Muirhead R J. *Aspects of muitivariate statitical theory* [M]. New York: Wiley, 1982.

[4] Zhu Daoyuan, Wu Chengou, Qin Weiliang. *Multivariate statistical analysis and the SAS software*[M]. Nanjing: Southeast University Press, 1999. (in Chinese)

拟 χ^2 分布与 Wishart 分布独立性

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摘　要　推广了卡方分布到拟卡方分布,证明了它的若干性质,并利用这些性质找到了 Wishart 分布内部及 2 个多元正态分布或矩阵正态分布之间独立的充要条件.

关键词　拟卡方分布, Wishart 分布, 独立性

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