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A Class of Star Extremal Circulant Graphs

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Abstract: The circular chromatic number and the fractional chromatic number are two generalizations of the ordinary chromatic number of a graph. A graph is called star extremal if its fractional chromatic number equals to its circular chromatic number (also known as the star chromatic number). This paper studies the star extremality of the circulant graphs whose generating sets are of the form $\{\pm 1, \pm k\}$.

Key words: circular chromatic number, fractional chromatic number, circulant graph, star extremal graph

Let k and d be two natural numbers such that $k \geqslant 2d$. A (k,d)-coloring of a graph $G=(V,E)$ is a mapping $c: V \to \{0,1,2,\cdots,k-1\}$, such that, for each edge $uv \in E$, $|c(u)-c(v)|_k \geqslant d$, where $|x|_k = \min\{|x|, k-|x|\}$. Observe that a $(k,1)$-coloring of a graph G is just an ordinary k-coloring of G. We say G is (k,d)-colorable if there exists a (k,d)-coloring of G. The circular chromatic number (also known as the star chromatic number which was first introduced by Vince[1]) $\chi_c(G)$ is the infimum of the ratios k/d for which there exist (k,d)-colorings of G. For a different but equivalent definition of the circular chromatic number, we refer the readers to Ref. [2]. The circular chromatic number is a natural generalization of the ordinary chromatic number and can be viewed as a refinement of the ordinary chromatic number. The concept of circular chromatic number of a graph has been studied in many papers[1,3-6].

Another generalization of the ordinary chromatic number is the fractional chromatic number of a graph G. A mapping c from the collection Γ of independent sets of a graph G to the interval $[0,1]$ is a fractional coloring if for every vertex x of G, we have $\sum_{S \in \Gamma, \text{s.t.}\, x \in S} c(S) = 1$. The value of a fractional coloring c is $\sum_{S \in \Gamma} c(S)$. The fractional chromatic number $\chi_f(G)$ of G is the infimum of the values of fractional colorings of G. For equivalent definitions of the fractional chromatic number, see Ref. [7]. For any graph G, it is well known that[8]

$$\max\left\{\omega(G), \frac{|V(G)|}{\alpha(G)}\right\} \leqslant \chi_f(G) \leqslant \chi_c(G) \leqslant \lceil \chi_c(G) \rceil = \chi(G)$$

where $\alpha(G)$ is the independence number of G. A graph G is called star extremal if $\chi_c(G) = \chi_f(G)$, that is the equality holds in the second inequation in the above inequation. This notion of star extremality of graphs was first introduced by Gao and Zhu when they studied the chromatic number and the circular chromatic number of the lexicographic product of graphs in Ref. [9]. Let p be a positive integer and let S be a subset of $\{1,2,\cdots,p-1\}$ such that $i \in S$ implies $p-i \in S$. For brevity, we write $-i$ for $p-i$. The circulant graph $G(p,S)$ has vertices $0,1,2,\cdots,p-1$ and i is adjacent to j if and only if $i-j \in S$, where subtraction is carried out modulo p. It is known and easy to prove that if a graph G is vertex transitive, then $\chi_f(G) = \frac{|V(G)|}{\alpha(G)}$, where $\alpha(G)$ is its independence number. Since any circulant graph $G = G(p,S)$ is vertex transitive, we have $\chi_f(G) = \frac{p}{\alpha(G)}$. Thus, to prove that $\chi_c(G) = \chi_f(G)$ for a circulant graph $G = G(p,S)$, it is sufficient to prove that $\chi_c(G) = \frac{p}{\alpha(G)}$.

Give a circulant graph $G = G(p,S)$ and an integer l, we let $\lambda_l(G) = \min\{|li|_p : i \in S\}$ and let $\lambda(G) = \max\{\lambda_l(G) : l = 1,2,\cdots\}$, where the multiplications li are carried out modulo p.

Lemma 1[9] Suppose G is a circulant graph. Then $\lambda(G) \leqslant \alpha(G)$. Moreover, if $\lambda(G) = \alpha(G)$ then G is star extremal.

Using lemma 1, Refs. [8] and [9] obtained some star extremal circulant graphs. In this paper, we give an improvement of the following theorem.

Theorem 1[9] Suppose $G = G(p,S)$ is a

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circulant graph and $|S| = 4$.

1) If $S = \{\pm 1, \pm k\}$, k is odd and $p > (k(k-3)+2)r/2$, where r is the unique number $0 \leqslant r < k$ satisfying $r = p(\mathrm{mod}\, k)$, then $G = G(p,S)$ is star extremal.

2) If $S = \{\pm 1, \pm k\}$, k is even and $p > k(k-1)$, then $G = G(p,S)$ is star extremal.

Theorem 2 Suppose $G = G(p,S)$ is a circulant graph and $|S| = 4$. If $S = \{\pm 1, \pm k\}$, k is odd and $p > (k^2 - 1)/2$, then $G = G(p,S)$ is star extremal.

Proof If p is even then G is bipartite. In this case G is clearly star extremal. So we assume that p is odd. We first show that $\alpha(G) = (p-k)/2$.

Let A be a maximum independent set of G. Since i is adjacent to $i+1$ for any i, it follows that if $i \in A$ then $i-1, i+1 \notin A$. If $I = \{i, i+2, \cdots, i+2t-2\} \subseteq A$(addition modulo p) and $i-2, i+2t \notin A$, then we call I a segment of A and t is called the length of the segment I. If A has only one segment, it is evident that the length of this segment is at most $(p-k)/2$. In this case, $|A| = (p-k)/2$. To prove that $\alpha(G) = (p-k)/2$, it is sufficient to prove that the cardinality of every independent set with at least two segments is at most $(p-k)/2$. Choose A among all maximum independent sets of G with at least two segments such that the length of the maximum segment of A is maximum.

Now we show that if $k \geqslant 5$ then the length of each segment I of A is less than or equal to $(k-3)/2$. Suppose to the contrary, let I be a segment of A with $(p-k)/2 > |I| = t > (k-3)/2$. Without loss of generality, we can assume $I = \{0, 2, \cdots, 2t-2\}$. Clearly $2t$ and $2t-1 \notin A$. Since $t \geqslant (k-1)/2$, $2t+1-k \in A$. It follows that $2t+1 \notin A$ and $2t-k \notin A$. If $2t+k \notin A$ then $A \cup \{2t\}$ is an independent set of G. This contradicts the fact that A is a maximum independent set of G. If $2t+k \in A$ then $A' = A \cup \{2t\} \setminus \{2t+k\}$ is also a maximum independent set of G with the length of its maximum segment greater than that of A. This contradicts the choice of A. Thus for each segment I of A, we have $|I| \leqslant (k-3)/2$, this implies that every $k+1$ consecutive vertices of G contains at most $(k-1)/2$ vertices of A.

Let $p = m(k+1) + r(1 \leqslant r \leqslant k)$, where r is odd. Since A has at least two segments, there exist two consecutive vertices $i, i+1 \notin A$, therefore $|A \cap \{i, i+1, \cdots, i+r-1\}| \leqslant \frac{r-1}{2}$. Thus $|A| \leqslant m\frac{k-1}{2} + \frac{r-1}{2}$. Since $p > \frac{k^2-1}{2}$, we have $m > \frac{k-1}{2}$. It follows that $|A| \leqslant m\frac{k-1}{2} + \frac{r-1}{2} \leqslant \frac{p-k}{2}$. Therefore $\alpha(G) = \frac{p-k}{2}$.

If $k = 3$, then choose A among all maximum independent sets of G such that the maximum segment of A is maximum. Without loss of generality, let $I = \{0, 2, \cdots, 2t-2\}$ be a maximum segment of A. Clearly $t \geqslant 1 > (k-3)/2$. If $t < (p-k)/2$, then, by the same argument in the proof of case $k \geqslant 5$, the same contradiction can be derived. Thus $t \geqslant (p-k)/2$. It follows that $A = I$ and $\alpha(G) = \frac{p-k}{2}$.

It is a routine to check that $\lambda_{\frac{p+k}{2}}(G) = \frac{p-k}{2}$. Hence $\lambda(G) \geqslant \lambda_{\frac{p+k}{2}}(G) = \frac{p-k}{2} = \alpha(G)$. By lemma 1, $\chi_c(G) = \chi_f(G)$ and G is star extremal.

Note that $(k^2-1)/2 \leqslant k(k-1)$, combine theorem 7 in Ref. [9] with the above theorem, the following corollary holds clearly.

Corollary 1 Let $G = G(p,S)$ be a circulant graph with $S = \{\pm 1, \pm k\}$. If $p > k(k-1)$ then G is star extremal.

Theorem 3 Suppose $S = \{\pm 1, \pm k\}$ and $G = G(p,S)$, p and k are odd with $p < (k^2-1)/2$. Let $p = m(k+1) + r, (0 \leqslant r \leqslant k)$, write $p = 2t+1$ and $k = 2s+1$. If $2m+1 \geqslant s+1-r \geqslant 0$, then $\alpha(G) = t-m$ and G is star extremal.

Proof From the proof of theorem 2, if $p < (k^2-1)/2$, $\alpha(G) \leqslant m\frac{k-1}{2} + \frac{r-1}{2}$, then $\alpha(G) \leqslant t-m$. On the other hand $\lambda_{t-m} = \min\{|t-m|_p, |(t-m)k|_p\} = \min\{t-m, |(t-m)(2s+1)|_p\}$. $|(t-m)(2s+1)|_p = |t(2s+2) - t - m(2s+1)|_p = |2t(s+1) + (s+1) - (s+1) + r - (t-m)|_p$.

Since $2m+1 \geqslant s+1-r \geqslant 0$, $|p - (t-m) + r - (s+1)|_p \geqslant t-m$. Thus $\lambda(G) = \lambda_{t-m} = t-m = \alpha(G)$ and G is star extremal.

Remark Suppose p, k are odd, and $S = \{\pm 1, \pm k\}$. From theorem 2, we see that if $p > \frac{k^2-1}{2}$ then $\alpha(G(p,S)) = \frac{p-k}{2}$. But, when $2k < p \leqslant \frac{k^2-1}{2}$, it seems difficult to determine its independence number. However, theorem 3 shows that if $G(p,S)$ satisfies $p < \frac{k^2-1}{2}$ and an additional condition, then $\alpha(G(p,S)) > \frac{p-k}{2}$ and $G(p,S)$ is still star extremal.

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一类具有 Star Extremal 性质的循环图

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摘　要　圆色数和分式色数是图的点色数的 2 个推广.当图的圆色数等于分式色数时,称此图是 star extremal.本文研究了生成集为$\{\pm 1, \pm k\}$具有 star extremal 特征的循环图.

关键词　圆色数，分式色数，循环图，star extremal 图

中图分类号　O157.5