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Nonhomogeneous(H, Q)-Process: The Backward and Forward Equations *

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Abstract: As for the backward and forward equation of nonhomogeneous(H, Q)-processes, we proof them in a new way. On the base of that, this paper gives the direct computational formal for one dimensional distribution of the nonhomogeneous(H, Q)-process.

Key words: nonhomogeneous(H, Q)-process, backward and forward equations, one-dimensional distribution

1997, Professor Hou, a famous mathematician, put forward the Markov skeleton processes[1], which are with a series of jump times with Markov property (not ruling out any other jump times). They spread the Markov processes very much, and likely include all kinds of those mixed systems, being followed with interest. This kind of processes has arouse widespread interests of scholars in China and many other countries. As for a kind of important Markov skeleton processes, homogeneous(H, Q)-processes has been widely studied. And until now, many magnificent achievements, have been made[1-5], and laid a preliminary theoretic foundation for them. While the nonhomogeneous property made this kind of process become more generality. The backward and forward equations of the nonhomogeneous(H, Q)-process have been obtained in Ref.[6], by turning one-dimensional nonhomogeneous(H, Q)-process into two-dimensional homogeneous(H, Q)-process and then using the backward and forward equations of the homogeneous(H, Q)-process. While in this paper, we proof them in another way ,and then discuss the one-dimensional distributions of them.

1 Nonhomogeneous(H, Q)-Process

Let (Ω, F, P) be a complete probability space, (E, ε)be a polish space. And $X = \{X(t,\omega), 0 \leqslant t < \tau(\omega)\}$ is a right-continuous and left-limit existing stochastic process defined on the (Ω, F, P), with its values in the (E, ε).

Definition 1 The stochastic process $X = \{X(t,\omega), 0 \leqslant t < \tau(\omega)\}$ is called the nonhomogeneous (H, Q)-process, if there exists a series of Markov times $\{\tau_n\}_{n \geqslant 0}$ satisfying the following:

(i) $0 = \tau_0 \leqslant \tau_1 \leqslant \tau_2 \leqslant \cdots, \tau = \lim\limits_{n\to\infty} \tau_n \qquad \mathrm{P-a.e}$

(ii) $E[X(\tau_n + t) \in A, \tau_{n+1} - \tau_n > t \mid X(\tau_n), \tau_n, X(\tau_{n-1}), \tau_{n-1}, \cdots, X(0)] =$
$E[X(\tau_n + t) \in A, \tau_{n+1} - \tau_n > t \mid X(\tau_n)] = h^{(n+1)}(t, X(\tau_n), A) \qquad n \geqslant 0, t \geqslant 0, A \in \varepsilon$

(iii) $E[X(\tau_{n+1}) \in A, \tau_{n+1} - \tau_n \leqslant t \mid X(\tau_n), \tau_n, X(\tau_{n-1}), \tau_{n-1}, \cdots, X(0)] =$
$E[X(\tau_{n+1}) \in A, \tau_{n+1} - \tau_n \leqslant t \mid X(\tau_n)] = q^{(n+1)}(t, X(\tau_n), A) \qquad n \geqslant 0, t \geqslant 0, A \in \varepsilon$

where for fixed A, $h^{(n)}(t,x,A)$, and $q^{(n)}(t,x,A)$ are measured functions of two variables; for fixed t and x, $h^{(n)}(t,x,A)$ and $q^{(n)}(t,x,A)$ are quasi-distributions on (E, ε).

Let $q^{(n)}(x,A) = \lim\limits_{t\to\infty} q^{(n)}(t,x,A)$, then by (iii), we know that $\{X(\tau_n)\}_{(n \geqslant 0)}$ is the nonhomogeneous Markov process with the stationary transition probabilities $\{q^{(n)}(x,A), x \in E, A \in \varepsilon\}$.

2 Backward and Forward Equations of the Nonhomogeneous (H, Q)-Process

Let $u_E \triangleq \{R \mid R(x,A)\}$ be non-negative function defined on $E \times \varepsilon$, and for fixed A, $R(x,A)$ is ε-measurable, and for fixed X, $R(x,A)$ is non-negative measure on $E \times \varepsilon\}$. Define the multiplication in u_E as follows: $\forall R, S \in u_E$,

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$$R\cdot S(x,A)\triangleq\int_E R(x,\mathrm{d}y)S(y,A)\qquad x\in E,A\in\varepsilon$$

Lemma 1 $\forall A\in\varepsilon,t\geqslant 0,n\geqslant 0$, then

$$E[X(t)\in A,\tau_n\leqslant t<\tau_{n+1}\mid X(\tau_n),\tau_n,X(0)]=h^{(n+1)}(t-\tau_n,X(\tau_n),A)\cdot 1_{\{\tau_n\leqslant t\}}\qquad \text{P-a.e}\quad (1)$$

Proof See lemma 2.1 in Ref.[2].

Let $q^{(n,k)}(t,x,A)=P(X(\tau_{n+k})\in A,\tau_{n+k}-\tau_n\leqslant t\mid X(\tau_n)=x)$, $\forall A\in\varepsilon,t\geqslant 0,x\in E,n\geqslant 0,k\geqslant 0$.

Lemma 2 $\forall A\in\varepsilon,t\geqslant 0,x\in E$, then

$$q^{(n,k)}(t,x,A)=q^{*(n,k)}(t,x,A)\qquad (2)$$

where

$$\begin{cases}q^{*(n,0)}(t,x,A)\triangleq\delta_A(x)\\ q^{*(n,1)}(t,x,A)=q^{(n+1)}(t,x,A)\\ q^{*(n,k)}(t,x,A)\triangleq\int_E\int_0^t q^{*(n+1,k-1)}(t-s,y,A)q^{(n+1)}(\mathrm{d}s,x,\mathrm{d}y)\end{cases}$$

Proof When $k=1,\forall n(n\geqslant 0)$, then

$$P(X(\tau_{n+1})\in A,\tau_{n+1}-\tau_n\leqslant t\mid X(\tau_n)=x)=q^{(n+1)}(t,x,A)=q^{*(n,1)}(t,x,A)$$

Assume that when $n=k$, (2) holds for any $n\geqslant 0$, then for $n=k+1$,

$$\begin{aligned}&P(X(\tau_{n+k+1})\in A,\tau_{n+k+1}-\tau_n\leqslant t\mid X(\tau_n)=x)=\\ &E[E[X(\tau_{n+k+1})\in A,\tau_{n+k+1}-\tau_n\leqslant t\mid X(\tau_{n+1}),\tau_{n+1},X(\tau_n)]\mid X(\tau_n)=x]=\\ &\int_\Omega E[X(\tau_{n+k+1})\in A,\tau_{n+k+1}-\tau_{n+1}+\tau_{n+1}-\tau_n\leqslant t\mid X(\tau_{n+1}),\tau_{n+1},X(\tau_n)]\cdot 1_{\{\tau_{n+1}-\tau_n\leqslant t\}}\cdot\\ &P(\mathrm{d}\omega\mid X(\tau_n)=x)=\int_E\int_0^t E[X(\tau_{n+k+1})\in A,\tau_{n+k+1}-\tau_{n+1}\leqslant t-s\mid X(\tau_{n+1})=y]\cdot\\ &P(X(\tau_{n+1})\in\mathrm{d}y,\tau_{n+1}-\tau_n\in\mathrm{d}s\mid X(\tau_n)=x)=\int_E\int_0^t q^{*(n+1,k)}(t-s,y,A)\cdot\\ &q^{(n+1)}(\mathrm{d}s,x,\mathrm{d}y)=q^{*(n,k+1)}(t,x,A)\end{aligned}$$

Let

$$q_\lambda^{(n,k)}(x,A)=\int_0^\infty \mathrm{e}^{-\lambda t}\mathrm{d}q^{(n,k)}(t,x,A),\ q_\lambda^{(n)}(x,A)=\int_0^\infty \mathrm{e}^{-\lambda t}\mathrm{d}q^{(n)}(t,x,A)\qquad x\in E,A\in\varepsilon$$

Lemma 3

$$q_\lambda^{(n,0)}(x,A)=\delta_A(x)\qquad (3)$$

$$q_\lambda^{(n,k)}(x,A)=\int_E\cdots\int_E q_\lambda^{(n+1)}(x,\mathrm{d}y_1)q_\lambda^{(n+2)}(y,\mathrm{d}y_2)\cdots q_\lambda^{(n+k)}(y_{k-1},A)\qquad (4)$$

Proof Using Laplace-Stieltjes transformation, we have the above results immediately.

Let

$$P(t,x,A)\triangleq P(X(t)\in A\mid X(0)=x)\qquad x\in E,A\in\varepsilon$$
$$P^{(n)}(t,x,A)\triangleq P(X(\tau_n+t)\in A\mid X(\tau_n)=x)\qquad x\in E,A\in\varepsilon$$

Let $P_\lambda(x,A),P_\lambda^{(n)}(x,A)$ be the Laplace transformation of $P(t,x,A),P^{(n)}(t,x,A)$, respectively.

Let

$$h_\lambda^{(n)}(x,A)=\int_0^\infty \mathrm{e}^{-\lambda t}h^{(n)}(t,x,A)\mathrm{d}t\qquad x\in E,A\in\varepsilon$$

$$h_\lambda(x,A)=\int_0^\infty \mathrm{e}^{-\lambda t}h(t,x,A)\mathrm{d}t,\ q_\lambda(x,A)=\int_0^\infty \mathrm{e}^{-\lambda t}\mathrm{d}q(t,x,A)\qquad x\in E,A\in\varepsilon$$

Theorem 1 $\{P_\lambda^{(n)}(x,A),n\in Z_+,x\in E,A\in\varepsilon\}$ is the minimal non-negative solution of the following non-negative equation

$$X^{(n)}(x,A)=\int_E q_\lambda^{(n+1)}(x,\mathrm{d}y)\cdot X^{(n+1)}(y,A)+h_\lambda^{(n+1)}(x,A)\qquad (5)$$

i.e.

$$P_\lambda^{(n)}(x,A)=\Big(\sum_{k=1}^\infty\Big(\prod_{m=1}^k Q_{n+m}\Big)\cdot H_{n+k+1}+H_{n+1}\Big)(x,A)\qquad (6)$$

In particular,

$$P_\lambda(x,A) = P_\lambda^{(0)}(x,A) = \sum_{k=0}^{\infty}\Big(\prod_{m=0}^{k} Q_m\Big)H_{k+1}(x,A) \tag{7}$$

where $Q_0 = (\delta_A(x)) \in u_E, H_m = (h_\lambda^{(m)}(x,A)) \in u_E, Q_m = (q_\lambda^{(m)}(x,A)) \in u_E, (m \geqslant 1)$.

Proof $P(X(\tau_n + t) \in A \mid X(\tau_n) = x) = P(X(\tau_n + t) \in A, \tau_{n+1} > \tau_n + t \mid X(\tau_n) = x) +$

$$P(X(\tau_n + t) \in A, \tau_{n+1} \leqslant \tau_n + t \mid X(\tau_n) = x) = h^{(n+1)}(t,x,A) + \sum_{k=1}^{\infty} P(X(\tau_n + t) \in A, \tau_{n+k} \leqslant \tau_n + t < \tau_{n+k+1} \mid X(\tau_n) = x)$$

Using lemma 1, we have

$$\begin{aligned}
&P(X(\tau_n + t) \in A, \tau_{n+k} \leqslant \tau_n + t < \tau_{n+k+1} \mid X(\tau_n) = x) = \\
&E[E[X(\tau_n + t) \in A, \tau_{n+k} \leqslant \tau_n + t < \tau_{n+k+1} \mid X(\tau_{n+k}), \tau_{n+k}] \mid X(\tau_n) = x] = \\
&\int_\Omega h^{(n+k+1)}(t - (\tau_{n+k} - \tau_n), X(\tau_{n+k}), A) \cdot 1_{\{\tau_{n+k} - \tau_n \leqslant t\}} P(\mathrm{d}\omega \mid X(\tau_n) = x) = \\
&\int_E \int_0^t h^{(n+k+1)}(t - s, y, A) \cdot P(X(\tau_{n+k}) \in \mathrm{d}y, \tau_{n+k} - \tau_n \in \mathrm{d}s \mid X(\tau_n) = x) = \\
&\int_E \int_0^t h^{(n+k+1)}(t - s, y, A) \cdot q^{(n,k)}(\mathrm{d}s, x, \mathrm{d}y)
\end{aligned}$$

Then using Laplace-Stieltjes transformation, we have

$$\begin{aligned}
P_\lambda^{(n)}(x,A) &= \sum_{k=1}^{\infty}\int_E q_\lambda^{(n,k)}(x,\mathrm{d}y) \cdot h_\lambda^{(n+k+1)}(y,A) + h_\lambda^{(n+1)}(x,A) = \\
&\Big(\sum_{k=1}^{\infty}\Big(\prod_{m=1}^{k} Q_{n+m}\Big) \cdot H_{n+k+1} + H_{n+1}\Big)(x,A)
\end{aligned}$$

where we used lemma 3.

Theorem 2 If for any $\lambda > 0$ and $n \geqslant 1$, there exists $\hat{Q}_n = (\hat{q}_\lambda^{(n)}(x,A)) \in u_E$, satisfying $H_{(n+1)} \cdot \hat{Q}_n = Q_n \cdot H_{n+1}$, i.e

$$\int_E h_\lambda^{(n+1)}(x,\mathrm{d}y)\hat{q}_\lambda^{(n)}(y,A) = \int_E q_\lambda^{(n)}(x,\mathrm{d}y)h_\lambda^{(n+1)}(y,A) \qquad x \in E, A \in \varepsilon \tag{8}$$

Then $\{P_\lambda^{(n)}(x,A), n \in Z_+, x \in E, A \in \varepsilon\}$ is the minimal non-negative solutions of the following non-negative equation,

$$X^{(n)}(x,A) = \int_E X^{(n+1)}(x,\mathrm{d}y)\hat{q}^{(n+1)}(y,A) + h_\lambda^{(n+1)}(x,A)(n \in Z_+, x \in E, A \in \varepsilon) \tag{9}$$

i.e

$$P_\lambda^{(n)}(x,A) = \sum_{k=1}^{\infty}\Big(H_{n+k+1} \cdot \prod_{m=1}^{k} \hat{Q}_{n+m} + H_{n+1}\Big)(x,A) \tag{10}$$

In particalar,

$$P_\lambda(x,A) = P_\lambda^{(0)}(x,A) = \sum_{k=0}^{\infty}\Big(H_{k+1} \cdot \prod_{m=0}^{k} \hat{Q}_m\Big)(x,A) \tag{11}$$

where $\hat{Q}_0 = Q_0 = (\delta_A(x)) \in u_E$.

Proof Using (8) and by substitution method, it is easy to proof that the minimal non-negative solutions of Eq.(9) and that of Eq.(5) are identical. And the solution is given by (10).

Definition 2 Eq.(5) and Eq.(9) are respectively called the backward equation and the forward equation of the non-homogeneous(H, Q)-process X.

As for process X, theorem 1 means that the Laplaces transformation $\{P_\lambda^{(n)}(x,A)\}$ of the transition probabilities is the minimal non-negative solution of the backward equation. While theorem 2 means that $\{P_\lambda^{(n)}(x, A)\}$ is also the minimal non-negative solution of the forward equation, if there exists the forward equation of the process X.

Corollary 1 If for any $\lambda > 0, n \geqslant 1, H_{n+1}$ has right inverse element in u_E, i.e. there exists $H_{n+1,\tau}^{-1} \in u_E$ satisfying $H_{n+1} \cdot H_{n+1,\tau}^{-1}(x,A) = \delta_A(x)$, then there exists the forward Eq.(9), where

$$\hat{Q}_n = H_{n+1,\tau}^{-1} \cdot Q_n \cdot H_{n+1} \quad n \geqslant 1.$$

Proof Noting that the multiplication in u_E satisfies the associative law, and using theorem 2, we have the above result.

Till now, we have gone round the indirectly method used in Ref.[6], and have directly proved the backward and forward equations of the non-homogeneous(H,Q)-process. Compared with the former, the proof in this paper looks not only succinet, but also direct and clear. And so, it makes us more directly understand and grasp the backward and forward equations of the nonhomogeneous(H,Q)-process.

3 One-dimensional Distribution of the Nonhomogeneous(H,Q)-Processes

Let $\pi(\mathrm{d}x)$ be the initial distribution of the nonhomogeneous(H,Q)-process $X=\{X(t,\omega),0\leqslant t<\tau(\omega)\}$, and

$$P(t,A) \triangleq P(X(t)\in A) \quad (t\geqslant 0, A\in\varepsilon), \quad P_\lambda(A) \triangleq \int_0^\infty \mathrm{e}^{-\lambda t}P(t,A)\mathrm{d}t \quad (\lambda>0) \tag{12}$$

Theorem 3 (i) $\forall \lambda>0, A\in\varepsilon$

$$P_\lambda(A) = \int_E \Big(\sum_{k=0}^{\infty}\Big(\prod_{m=0}^{k} Q_m\Big)\cdot H_{k+1}(x,A)\Big)\pi(\mathrm{d}x) \tag{13}$$

(ii) If there exist the forward Eq.(9) exist, then for any $\lambda>0$ and $A\in\varepsilon$, we have

$$P_\lambda(A) = \int_E \Big(\sum_{k=0}^{\infty}\Big(H_{k+1}\cdot\prod_{m=0}^{k}\hat{Q}_m\Big)(x,A)\Big)\pi(\mathrm{d}x) \tag{14}$$

where $\hat{Q}_0 = Q_0 = (\delta_A(x)) \in u_E$.

Proof Using (7),(11) and (12), we have the above result.

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非齐次(H,Q)-过程：向前、向后方程

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摘　要　本文对非齐次(H,Q)-过程定义的已有成果:向前、向后方程给予了新的证明;在此基础上本文给出了非齐次(H,Q)-过程的一维分布的具体计算公式.

关键词　非齐次(H,Q)-过程,向前、向后方程,一维分布

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