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Neighborhood Union of Essential Sets and Hamiltonicity of Claw-Free Graphs*

Xu Xinping**

(School of Mathematics and Computer Science, Nanjing Normal University, Nanjing 210097, China)
(Department of Mathematics and Computer Science, Jiangsu Institute of Education, Nanjing 210013, China)

Abstract: Let G be a graph, an independent set Y in G is called an essential independent set (or essential set for simplicity), if there is $\{y_1, y_2\} \subseteq Y$ such that dist $(y_1, y_2) = 2$. In this paper, we will use the technique of the vertex insertion on l-connected ($l = k$ or $k+1, k \geqslant 2$) claw-free graphs to provide a unified proof for G to be hamiltonian or 1-hamiltonian, the sufficient conditions are expressed by the inequality concerning $\sum_{i=0}^{k} |N(Y_i)|$ and $n(Y)$ for each essential set $Y = \{y_0, y_1, \cdots, y_k\}$ of G, where $Y_i = \{y_i, y_{i-1}, \cdots, y_{i-(b-1)}\} \subseteq Y$ for $i \in \{0,1,\cdots,k\}$ (the subscriptions of y_j's will be taken modulo $k+1$), b $(0 < b < k+1)$ is an integer, and $n(Y) = |\{v \in V(G): \text{dist}(v, Y) \leqslant 2\}|$.

Key words: hamiltonicity, claw-free graph, neighborhood union, vertex insertion, essential set

In this paper, the terminology and notation not defined will follow Ref. [1], and we consider simple finite graphs only. A graph G is called claw-free if it has no induced subgraph isomorphic to $K_{1,3}$. A cycle C of G is called a hamiltonian cycle if C is a spanning cycle. A graph G is called hamiltonian if there exists a hamiltonian cycle in G; 1-hamiltonian graph if $G - \{w\}$ is hamiltonian for any $w \in V(G)$. An independent set Y in G is called an essential independent set (or essential set for simplicity) if there is $\{y_1, y_2\} \subseteq Y$ such that $\text{dist}(y_1, y_2) = 2$.

Let $t > 1$ be an integer. Denote

$$I_t(G) = \{Y | Y \text{ is an independet set of } G,\ |Y| = t\}$$

$$I_t^{(e)}(G) = \{Y | Y \text{ is an essential set of } G, |Y| = t\}$$

Let G be connected, $Y \subseteq V(G)$, $|Y| = t$, and $v \in V(G)$. Denote $\text{dist}(v, Y) = \min_{y \in Y}\{\text{dist}(v, y)\}$ (where dist (v, y) stands for the distance between v and y), $N_i(Y) = \{v \in V(G): \text{dist}(v, Y) = i\}$ $(i = 0,1,2,\cdots)$, and $n(Y) = |N_0(Y) \cup N_1(Y) \cup N_2(Y)| = |\{v \in V(G): \text{dist}(v, Y) \leqslant 2\}|$.

For each $i \in \{0,1,2,\cdots,t\}$, denote

$$S_i(Y) = \{v \in V(G): |N(v) \cap Y| = i\}$$

Clearly

$$N(Y) = N_1(Y) = \bigcup_{i=1}^{t} S_i(Y)$$

and

$$n(Y) = |V(G) \setminus \bigcup_{l>2} N_l(Y)| \leqslant |V(G)|$$

In this paper, we will prove the following new results (theorems 1 and 2) by using the vertex inserting lemmas introduced in Ref. [4]. In theorems 1 and 2, we always assume that $Z = \{z_0, z_1, \cdots, z_k\} \in I_{k+1}(G)$ which is an order set, b be an integer and $0 < b < k+1$. Set

$$Z_i = \{z_i, z_{i-1}, \cdots, z_{i-(b-1)}\} \subseteq Z$$

for $i \in \{0,1,\cdots,k\}$ (where the subscriptions of z_j's will be taken modulo $k+1$).

Theorem 1 Let G be a k-connected claw-free graph with $k \geqslant 2$; b an integer $(0 < b < k+1)$. If

$$\sigma_b(Z) = \sum_{i=0}^{k} |N(Z_i)| > b(n(Z) - 1)$$

for each $Z \in I_{k+1}^{(e)}(G)$, then G is hamiltonian.

In theorem 1, when $b = 2$, we have the following result.

Corollary 1 Let G be a k-connected claw-free graph with $k \geqslant 2$, and $|V(G)| = n$. If

$$|N(u) \cup N(v)| > \frac{2(n-1)}{k+1}$$

for any $\{u, v\} \subset V(G)$, and dist $(u, v) = 2$, then G is hamiltonian.

Theorem 2 Let G be a $(k+1)$-connected claw-free graph with $k \geqslant 2$; b an integer $(0 < b < k+1)$. If

$$\sigma_b(Z) = \sum_{i=0}^{k} |N(Z_i)| > bn(Z)$$

for each $Z \in I_{k+1}^{(e)}(G)$, then G is 1-hamiltonian.

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** Born in 1964, female, associate professor.

Theorem 3[5] Let G be a k-connected claw-free graph with $k \geqslant 2$. If

$$\sum_{z \in Z} d(z) > n(Z) - 1$$

for each $Z \in I_{k+1}^{(e)}(G)$, then G is hamiltonian.

Clearly, theorem 1 improves and generalizes theorem 3.

Finally, we will use, in additional, the following notations.

Sometimes, by a slight abuse of notation, we shall use the same letter for a subgraph (of G) and its vertex set, provided no ambiguity arises.

Let U and R be subgraphs of G (or subsets of $V(G)$), denote$N_R(U) = N(U) \cap R$.

Each cycle or path of G discussed in this paper will be assigned an orientation. Let B be a cycle or path of G, $\{x, y\} \subseteq V(B)$, denote by $B[x, y]$ the oriented (x, y)-path of B (where the orientation was taken from B), $B(x, y] = B[x, y] - \{x\}$, $B[x, y) = B[x, y] - \{y\}$, and $B(x, y) = B[x, y] - \{x, y\}$.

1 The Basic Lemmas

In this section, we always assume that G is a connected non-hamiltonian graph and C is a maximal cycle of G (i.e., there is no cycle C' in G, such that $V(C) \subset V(C')$), and H is a component of $G - V(C)$. Assume also $\{v_1, v_2, \cdots, v_k\} \subseteq N_C(H)$ and $v_1, v_2, \cdots, v_k$ occur on C in the order of their indices. The subscriptions of v_i's will be taken modulo k. If $x \in V(C)$, denote by x^+ and x^- the successor and the predecessor of x along the orientation of C, respectively.

For each $i \in \{1, 2, \cdots, k\}$, a vertex $u \in C(v_i, v_{i+1})$ is called insertible[4], if there is some vertex $w \in C[v_{i+1}, v_i)$ such that $\{w, w^+\} \subseteq N(u)$. Otherwise u is called non-insertible.

Lemma 1[4] Let $u \in C(v_i, v_{i+1})$ for some $i \in \{1, 2, \cdots, k\}$. If all vertices in $C(v_i, u)$ are insertible, then $u \notin N_C(H)$. Therefore there exists a vertex in $C(v_i, v_{i+1})$, which is non-insertible.

By lemma 1, for each $i \in \{1, 2, \cdots, k\}$, let x_i be the first non-insertible vertex in $C(v_i, v_{i+1})$.

Lemma 2[4] ① If $u \in N_C(H)$, then $u^+ \notin N_C(H)$;

② If $u \in N(x_i) \cap C[v_{i+1}, v_i)$, then $u^+ \notin N(x_i)$.

Lemma 3[4] For $\{i, j\} \subseteq \{1, 2, \cdots, k\}$, if $y_i \in C(v_i, x_i]$, $y_j \in C(v_j, x_j]$, then

① There is no (y_i, y_j)-path Q with all its internal vertices not in $V(C)$;

② There is no $w \in C[y_i, y_j]$, such that $\{y_j w, y_i w^+\} \subseteq E(G)$.

Lemma 4[4] If $u \in N_C(H) \setminus \{v_1, v_2, \cdots, v_k\}$, $y \in \bigcup_{j=1}^{k} C(v_j, x_j]$, then $u^+ y \notin E(G)$.

In the remaining part of this paper, let $Y = \{y_0, y_1, \cdots, y_k\}$, where $y_0 \in V(H)$, $y_i \in C(v_i, x_i]$ for $i \in \{1, 2, \cdots, k\}$. Denote $J_Y = \bigcup_{i=1}^{k} C[y_i, v_{i+1}]$, $K_Y = V(G) \setminus J_Y$.

Lemma 5[4] $Y \in I_{k+1}(G)$, $K_Y \subseteq S_0(Y) \cup S_1(Y)$, $K_Y \cap N_0(Y) = \{y_0\}$.

A segment $C[z_1, z_2)(\subseteq C[y_t, v_{t+1}]$ $t \in \{1, 2, \cdots, k\})$ is called a CY-segment. If

① $C(z_1, z_2) \cap S_0(Y) = \varnothing$,;

② $z_1 \in N_2(Y) \cup Y$, $z_2 \in S_0(Y) \cup \{v_{t+1}^+\}$.

A CY-segment $C[z_1, z_2)$ is said to be simple if $C(z_1, z_2) \subseteq S_1(Y)$.

Lemma 6 Let $C[z_1, z_2)$, $(\subseteq C[y_t, v_{t+1}]$, $t \in \{1, 2, \cdots, k\})$ be a CY-segment. Then the following results hold.

① Let $M_i = N(y_i) \cap C(z_1, z_2)$, $(i \in \{0, 1, 2, \cdots, k\})$, then

$$M_t, M_{t-1}, \cdots, M_1, M_k, M_{k-1}, \cdots, M_{t+1}, M_0$$

(some of them may be empty) form consecutive subpaths of $C(z_1, z_2)$ which can have only their endvertices in common, and $|M_0| \leqslant 1$; $|M_i| \leqslant 1$ when $y_i = x_i$, and $i \in \{1, 2, \cdots, k\} \setminus \{t\}$.

② Let $Z_1 = C(z_1, z_2) \cap S_1(Y)$, $Z_2 = C(z_1, z_2) \setminus Z_1 = \{w_1, w_2, \cdots, w_h\}(h \geqslant 0)$, and $w_j \in S_{i_j}(Y)$., Then

(a) $|Z_1| + |Z_2| = |C[z_1, z_2)| - 1$;

(b) If G be a claw-free graph, and there is $y_q \in Y \setminus \{y_0\}$, such that $C(z_1, z_2) \subseteq S_1(Y) \cup (S_2(Y) \cap N(y_q))$, then $h = 1$.

Proof By lemma 6 in Ref. [4], ① and ②(a) hold. By lemmas 2, 3 and 4, and G be a claw-free graph, (b) holds.

In the following lemmas, we always choose $x_0 \in N_H(v_q)$ for some $q \in \{1, 2, \cdots, k\}$, let $X = \{x_0, x_1, \cdots, x_k\}$, and $Y = (X \setminus \{x_q\}) \cup \{v_q^+\}$.

Lemma 7[3] $Y \in I_{k+1}^{(e)}(G)$.

Lemma 8[5] Let G be a claw-free graph. Then

① $S_i(X) = \varnothing$ for $i \in \{2,3,,\cdots,k+1\}$, and $V(G) = S_0(X) \cup S_1(X)$;

② $S_i(Y) = \varnothing$ for $i \in \{3,4,\cdots,k+1\}$, and $V(G) = S_0(Y) \cup S_1(Y) \cup (S_2(Y) \cap N(v_q^{+}))$.

We always assume that b is an integer $(0 < b < k+1)$, $Y_i = \{y_i, y_{i-1}, \cdots, y_{i-(b-1)}\}$ $(\subseteq Y)$ for $i \in \{0,1,\cdots,k\}$, the subscriptions of y_j's will be taken modulo $k+1$.

Let $U \subseteq V(G)$. We always set

$$\sigma_b(U,Y) = \sum_{i=0}^{k} |N(Y_i) \cap U|$$

$$\sigma_b(Y) = \sigma_b(V(G),Y) = \sum_{i=0}^{k} |N(Y_i)|$$

By the definition of $\sigma_b(Y)$, it is not difficult to check that the following lemma holds.

Lemma 9 ① If $w \in S_1(Y)$, then $\sigma_b(\{w\},Y) = b$;

② If $w \in S_2(Y)$, then $\sigma_b(\{w\},Y) \leqslant 2b$.

Lemma 10 ① $\sigma_b(K_Y,Y) \leqslant b(|K_Y| - 1 - |\bigcup_{l>2}(N_l(Y) \cap K_Y)|)$;

② Let G be a claw-free graph, and $C[z_1,z_2](\subseteq C[y_t,v_{t+1}])$ is a CY-segment, then

$$\sigma_b(C[z_1,z_2),Y) \leqslant b|C[z_1,z_2)|$$

Proof ① By lemma 5, $K_Y \subseteq S_0(Y) \cup S_1(Y)$, and $K_Y \cap N_0(Y) = \{y_0\}$. So by lemma 9①,

$$\sigma_b(K_Y,Y) = b(|K_Y| - |K_Y \cap (N_0(Y) \cup N_2(Y))| - |\bigcup_{l>2}(N_l(Y) \cap K_Y)|) \leqslant b(|K_Y| - 1 - |\bigcup_{l>2}(N_l(Y) \cap K_Y)|)$$

so ① holds.

② If $C[z_1,z_2)$ be a simple CY-segment, by lemma 9①,

$$\sigma_b(C[z_1,z_2),Y) = b|C(z_1,z_2)| = b(|C[z_1,z_2)| - 1)$$

If $C[z_1,z_2)$ be non-simple CY-segment, by lemma 8②, $C(z_1,z_2) \subseteq S_1(Y) \cup (S_2(Y) \cap N(v_q^{+}))$. By lemma 6②, when $h = |Z_2| = 1$, thus by lemma 9②,

$$\sigma_b(C[z_1,z_2),Y) \leqslant b|Z_1| + 2b|Z_2| = b(|Z_1| + |Z_2| + 1) = b|C[z_1,z_2)|$$

so ② holds.

Lemma 11 Let G be a claw-free graph, then $\sigma_b(Y) \leqslant b(n(Y) - 1)$.

Proof We first prove two results.

(a) $\sigma_b(C[y_t,v_{t+1}],Y) \leqslant b(|C[y_t,v_{t+1}]| - |\bigcup_{l>2}(N_l(Y) \cap C[y_t,v_{t+1}])|)$.

In fact, for $t \in \{1,2,\cdots,k\}$, partition

$$C[y_t,v_{t+1}] \setminus \bigcup_{l>2}(N_l(Y) \cap C[y_t,v_{t+1}])$$

into s_t CY-segments

$$C[z_{11}^{(t)}, z_{12}^{(t)}), C[z_{21}^{(t)}, z_{22}^{(t)}), \cdots, C[z_{s_t 1}^{(t)}, z_{s_t 2}^{(t)})$$

Thus by lemma 10②, we have

$$\sigma_b(C[y_t,v_{t+1},Y] = \sum_{j=1}^{s_t} \sigma_b(C[z_{j1}^{(t)}, z_{j2}^{(t)}),Y) \leqslant \sum_{j=1}^{s_t} b|C[z_{j1}^{(t)}, z_{j2}^{(t)})| = b(|C[y_t,v_{t+1}]| - |\bigcup_{l>2}(N_l(Y) \cap C[y_t,v_{t+1}])|)$$

so (a) holds.

(b) $\sigma_b(J_Y,Y) \leqslant b(|J_Y| - |\bigcup_{l>2}(N_l(Y) \cap J_Y)|)$.

In fact, note that $J_Y = \bigcup_{t=1}^{k} C[y_t,v_{t+1}]$. By (a), we have

$$\sigma_b(J_Y,Y) = \sum_{t=1}^{k} \sigma_b(C[y_t,v_{t+1}],Y) \leqslant \sum_{t=1}^{k} b(|C[y_t,v_{t+1}]| - |\bigcup_{l>2}(N_l(Y) \cap C[y_t,v_{t+1}])|) = b(|J_Y| - |\bigcup_{l>2}(N_l(Y) \cap J_Y)|)$$

so (b) holds.

Now we prove the lemma. Note that $V(G) = J_Y \cup K_Y$. Thus by (b) and lemma 10①,

$$\sigma_b(Y) = \sigma_b(J_Y,Y) + \sigma_b(K_Y,Y) \leqslant b(|J_Y| - |\bigcup_{l>2}(N_l(Y) \cap J_Y)|) + b(|K_Y| - 1 - |\bigcup_{l>2}(N_l(Y) \cap K_Y)|) = b(|V(G) \setminus \bigcup_{l>2} N_l(Y)| - 1) = b(n(Y) - 1)$$

2 Proofs of the Theorems

Proof of theorem 1 By contradiction. Suppose that G is non-hamiltonian. Since G is a k-connected graph with $k \geqslant 2$, we may choose a longest cycle C of G, a component H of $G - V(C)$ and $\{v_1, v_2, \cdots, v_k\} \subseteq N_C(H)$. Suppose that $v_1, v_2, \cdots, v_k$ occur on C in the order of their indices. By lemma 1, for each $i \in \{1,2,\cdots,k\}$, let x_i be the first non-insertible vertex in $C(v_i,v_{i+1})$. Set $X = \{x_0, x_1, \cdots, x_k\}$, where $x_0 \in N_H(v_q)$ for some $q \in \{1,2,\cdots,k\}$. By lemma 8①, $V(G) = S_0(X) \cup S_1(X)$. Set $Y = (X \setminus \{x_q\}) \cup \{v_q^{+}\}$. Thus, by lemma 7, $Y \in I_{k+1}^{(e)}(G)$. For convenience, denote $Y = \{y_0, y_1, \cdots, y_k\}$.

On the other hand, by lemma 11, we have

$$\sigma_b(Y) = \sum_{i=0}^{k} |N(Y_i)| \leqslant b(n(Y) - 1)$$

a contradiction.

The following theorem will involve a graph G' other than G. In order to distinguish the notations such as $N(U)$, $N_j(X)$, K_X, $S_i(X)$, $n(X)$, $\sigma_b(X)$ introduced for G, we will simply add a prime to the notations with respect to G'. For example, $N'(U)$, $N'_j(X)$, etc.

Proof of Theorem 2 By contradiction. Suppose there exists $w \in V(G)$ such that $G' = G - \{w\}$ is non-hamiltonian. Since G is a $(k+1)$-connected graph with $k \geqslant 2$, we may choose a longest cycle C of G', a component H of $G' - V(C)$, and $\{v_1, v_2, \cdots, v_k\} \subseteq N'_C(H)$. Suppose that $v_1, v_2, \cdots, v_k$ occur on C in the order of their indices. By lemma 1, for each $i \in \{1, 2, \cdots, k\}$, let x_i be the first non-insertible vertex in $C(v_i, v_{i+1})$. Set $X = \{x_0, x_1, \cdots, x_k\}$, where $x_0 \in N(v_q)$ for some $q \in \{1, 2, \cdots, k\}$. Obviously G' is also claw-free graphs. By Lemma 8①, $V(G') = S'_0(X) \cup S'_1(X)$. Set $Y = (X \setminus \{x_q\}) \cup \{v_q^+\}$. By lemma 7, $Y \in I_{k+1}^{(e)}(G')$. Note that $G' = G - \{w\}$, it is easy to see that $Y \in I_{k+1}^{(e)}(G)$. For convenience, denote $Y = \{y_0, y_1, \cdots, y_k\}$.

On the other hand, note that G is a claw-free graph, $w \in S_0(Y) \cup S_1(Y) \cup S_2(Y)$. Set $\xi = 0$ if $w \in S_0(Y)$, $\xi = 1$ if $w \in S_1(Y) \cup S_2(Y)$; so $n'(Y) + \xi \leqslant n(Y)$. Thus by lemmas 11 and 9②, we have

$$\begin{aligned}\sigma_b(Y) &= \sum_{i=0}^{k} |N(Y_i)| = \sum_{i=0}^{k} |N'(Y_i)| + \\ &\sum_{i=0}^{k} |N'(Y_i) \cap \{w\}| \leqslant \\ &\sigma'_b(Y) + 2b\xi \leqslant b(n'(Y) - 1 + 2\xi) \leqslant \\ &b(n(Y) - 1 + < \xi) \leqslant bn(Y)\end{aligned}$$

a contradiction.

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本质集的邻域并和无爪图的哈密尔顿性

徐新萍

(南京师范大学数学与计算机科学学院,南京 210097)

(江苏省教育学院数学与计算机科学系,南京 210013)

摘 要 设 G 是一个图,G 的独立集 Y 称为本质集,如果存在$\{y_1, y_2\} \subseteq Y$,使得 $\mathrm{dist}(y_1, y_2) = 2$.本文利用插点方法,给出了关于 k 或$(k+1)$-连通$(k \geqslant 2)$无爪图 G 是哈密尔顿的或 1-哈密尔顿的统一的证明.2 个结果的充分条件是关于 $\sum_{i=0}^{k} |N(Y_i)|$ 与 $n(Y)$的不等式,这里 Y 是图 G 的任一本质集,对于 $i \in \{0, 1, \cdots, k\}$, $Y_i = \{y_1, y_{i-1}, \cdots, y_{i-(b-1)}\} \subseteq Y$($y_j$ 的下标将取模 $k+1$);b 是一个整数,且 $0 < b < k+1$;$n(Y) = |\{v \in V(G): \mathrm{dist}(v, Y) \leqslant 2\}|$.

关键词 哈密尔顿性,无爪图,邻域并,插点,本质集

中图分类号 O157.5