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Unified Solution Method of Rectangular Plate Elastic Bending

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Abstract: The bending of rectangular plate is divided into the generalized statically determinate bending and the generalized statically indeterminate bending based on the analysis of the completeness of calculating condition at the corner point. The former can be solved directly by the equilibrium differential equation and the boundary conditions of four edges of the plate. The latter can be solved by using the superposition principle. Making use of the recommended method, the bending of the plate with all kinds of boundary, such as simply supported edge, clamped edge, free edge, free corner point and pillar support corner point, can be solved under arbitrary loads, such as the loads on plate, the loads in plate edge, the load at free corner point, and when the plate edge and the pillar support corner point have settlement or when the plate edge has rotation. The method can organically unite the Navier solution and the Levy solution and has the advantages of rapid convergence and high precision.

Key words: bending of elastic thin plate, rectangular plate, unified solution method

The study of thin plate bending is one of the classical problems. This problem is not solved completely yet because of the variety of boundary conditions, the complexity of loads and the limitation of available solutions[1]. We began to study the unified solution method of rectangular plate bending in 1993 and since then the bending of rectangular plates with various boundary has been solved under arbitrary loads[2-14]. The unified solution method in this paper is a more general method which not only gives the essential idea for above bending solution, but also can solve the bending of plate when the plate edge and the pillar support corner point have settlement or when the plate edge has rotation[15].

1 Two Kinds of Rectangular Plate Bending

Rectangular plate bending can be divided into generalized statically determinate problem and generalized statically indeterminate problem, according to the completeness of the calculating condition at the corner point. The bending of the plate with pillar support corner point whose concentrated supporting force cannot be derived from the static equations of equilibrium is generalized statically indeterminate problem, the others are generalized statically determinate problem. Generalized statically determinate bending can be solved directly by the equilibrium differential equation and the boundary conditions of four edges. The generalized statically indeterminate bending can be solved by the superposition principle.

The deflection w of the plate should satisfy the equilibrium differential equation

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q(x,y)}{D} \qquad (1)$$

where $q(x,y)$ is the normal load on the plate; D is the flexural rigidity of the plate.

Moreover, the deflection should still satisfy the boundary conditions of four edges and corner point conditions. The corner points can be divided into three basic types.

1) End corner point of supported edge

Here, the supported edge may be clamped edge or simple supported edge, and the end corner point of supported edge may be the end point of one supported edge or the intersecting point of two supported edges. If the deflection w satisfies the displacement condition of the edge, it can satisfy the displacement condition of the corner point. After knowing the deflection w, the concentrated supporting force of the corner point can be calculated. The calculating condition of the corner point is complete because the corner point hasn't any restriction for the expression of w.

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2) Free corner point

The deflection of the corner point is that of the intersecting point of two adjacent free edges. The concentrated force of the corner point is zero or a concentrated load acting at the corner point. If the deflection expression itself satisfies the concentrated force condition of the corner point, the calculating conditions of the corner point must be complete.

3) Pillar support corner point

The corner point is the intersecting point of two free edges and supported with pillar. The calculating condition of this corner point can be divided into two types, according as whether the concentrated supporting force of the corner point can be got by static equilibrium condition or not. For the rectangular plate with one edge simply supported and one pillar support corner point or the plate with three pillar support corner points, whose concentrated supporting force at the corner point can be determined by the statically balanced conditions, if the deflection expression itself satisfies the displacement condition of the corner point that the deflection equals zero or a certain value, the calculating condition of this corner point is complete, because the concentrated supporting force will be derived from deflection expression which satisfies Eq. (1) and all boundary conditions. The displacement of the pillar support corner point, which has nothing to do with concentrated supporting force, only brings on rigid-body displacements of the plate, these characteristics are similar to statically determinate structure. When the concentrated supporting force of pillar support corner point can't be determined by the statically balanced conditions, such as the plate with four pillar support corner points, the plate with one edge simply supported and two pillar support corner points, the plates with one clamped edge and one or two pillar support corner points and the plates with two adjacent edges supported and one pillar support corner point, the calculating conditions of these corner points are not complete. Because only three statically balanced conditions are included in Eq.(1), the concentrated supporting force at the corner points is not derived from deflection expression which satisfies Eq.(1) and all boundary conditions. The displacement of the pillar support corner point, which is correlative with internal force in the plate and concentrated supporting force at the corner point, brings on bending deformation. These characteristics are similar to statically indeterminate structure.

2 Homogeneous Solution of Rectangular Plate Bending

The generalized statically determinate bending can be solved directly by the equilibrium differential equation and the boundary conditions of four edges. The expression of deflection is written as

$$w = w_1 + w_2 \tag{2}$$

where w_1 is a homogeneous solution of Eq.(1); w_2 is a particular solution of Eq.(1). The homogeneous solution w_1, which is related to the edge kind and the displacement of the end corner point of supported edge, must satisfy the condition that the concentrated force of free corner point is zero and the deflection condition of pillar support corner point. The expression w_1 is bi-directional single trigonometric series including eight undetermined coefficients. The trigonometric series must be whole orthogonal trigonometric function family, and its form must accord with bending deformation brought on by boundary conditions. For example, the values of two ends of the series waveform must equal zero for the plate with two opposite edges supported, the values of two ends don't equal zero for the plate with two opposite edges free and the value of one end is zero and of other is not zero for the plate with one edge supported and the opposite edge free. The deflection expression w_1 should reflect the deformation characteristic brought on by the displacements of end corner points of the supported edges. And the value of deflection w_1 is not zero at the free corner point. The rectangular plate of generalized statically determinate bending can be divided into seven types according to the boundary conditions.

2.1 Rectangular plates with four edges supported (Fig.1)

$$\begin{aligned} w_1 = {} & \sum_{m=1,2,3}^{\infty} (A_m \operatorname{sh}\alpha y + B_m \operatorname{ch}\alpha y + C_m \alpha y \operatorname{sh}\alpha y + \\ & D_m \alpha y \operatorname{ch}\alpha y)\sin\alpha x + \sum_{n=1,2,3}^{\infty} (E_n \operatorname{sh}\beta x + F_n \operatorname{ch}\beta x + \\ & G_n \beta x \operatorname{sh}\beta x + H_n \beta x \operatorname{ch}\beta x)\sin\beta y + \Delta_O + \frac{(\Delta_A - \Delta_O) y}{b} + \\ & \frac{(\Delta_B - \Delta_O) x}{a} + \frac{(\Delta_C - \Delta_A - \Delta_B + \Delta_O) xy}{ab} \end{aligned} \tag{3}$$

where $\alpha = \dfrac{m\pi}{a}$; $\beta = \dfrac{n\pi}{b}$; A_m, B_m, C_m, D_m, E_n, F_n, G_n and H_n are eight undetermined coefficients; Δ_O, Δ_A, Δ_B and Δ_C denote the displacements of O, A, B and C shown in Fig.1, respectively. The polynomial, whose coefficients are made of Δ_O, Δ_A, Δ_B and Δ_C, reflects

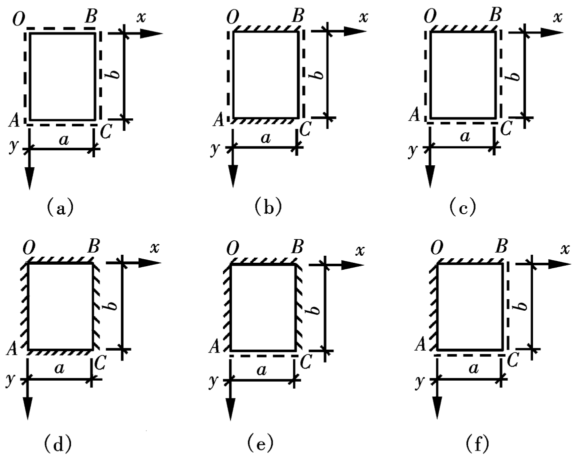

Fig. 1 Plates with four edges supported

the non-bending deformation characteristic of the plate brought on by the displacements of the end corner points of supported edges.

2.2 Rectangular plates with three edges supported and one edge free (Fig.2)

$$w_1 = \sum_{m=1,2,3}^{\infty} (A_m \mathrm{sh}\alpha y + B_m \mathrm{ch}\alpha y + C_m \alpha y \mathrm{sh}\alpha y + D_m \alpha y \mathrm{ch}\alpha y)\sin\alpha x + \sum_{n=1,3,5}^{\infty} (E_n \mathrm{sh}\lambda x + F_n \mathrm{ch}\lambda x + G_n \lambda x \mathrm{sh}\lambda x + H_n \lambda x \mathrm{ch}\lambda x)\sin\lambda y + \Delta_O + \frac{(\Delta_A - \Delta_O)y}{b} + \frac{(\Delta_B - \Delta_O)x}{a} + \frac{(\Delta_C - \Delta_A - \Delta_B + \Delta_O)xy}{ab} \quad (4)$$

where $\alpha = \dfrac{m\pi}{a}$; $\lambda = \dfrac{n\pi}{2b}$; $\sum\limits_{n=1,3,5} \sin\dfrac{n\pi y}{2b}$ is orthogonal from 0 to b. The corner point O, A, B and C are all the end corner points of supported edges.

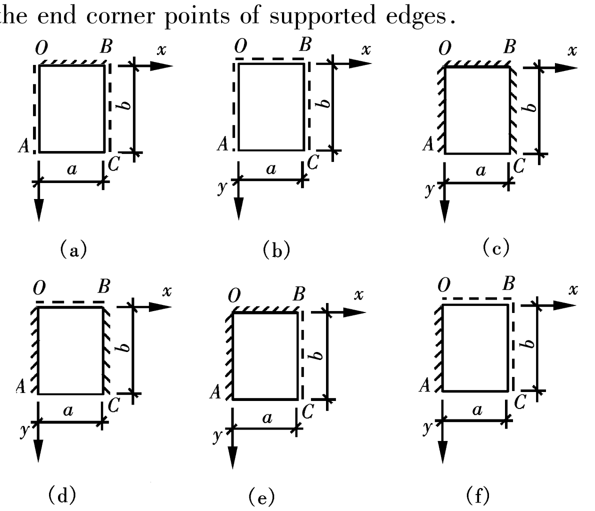

Fig. 2 Plates with three edges supported and one edge free

2.3 Rectangular plates with two opposite edges supported and two edges free (Fig.3)

$$w_1 = \sum_{m=1,2,3}^{\infty} (A_m \mathrm{sh}\alpha y + B_m \mathrm{ch}\alpha y + C_m \alpha y \mathrm{sh}\alpha y + D_m \alpha y \mathrm{ch}\alpha y)\sin\alpha x + (E_0 x + F_0 + G_0 x^2 + H_0 x^3) + \sum_{n=1,2,3}^{\infty} (E_n \mathrm{sh}\beta x + F_n \mathrm{ch}\beta x + G_n \beta x \mathrm{sh}\beta x + H_n \beta x \mathrm{ch}\beta x)\cos\beta y + \Delta_O + \frac{(\Delta_A - \Delta_O)y}{b} + \frac{(\Delta_B - \Delta_O)x}{a} + \frac{(\Delta_C - \Delta_A - \Delta_B + \Delta_O)xy}{ab} \quad (5)$$

where $\alpha = \dfrac{m\pi}{a}$; $\beta = \dfrac{n\pi}{b}$; the undetermined coefficients E_0, F_0, G_0 and H_0 are the E_n, F_n, G_n and H_n respectively when n equals zero. $\sum\limits_{n=0,1,2} \cos\beta y$ is a whole orthogonal trigonometric function family from 0 to b. The hyperbolic function corresponding to $n = 0$ must become a polynomial about x in order to ensure the integrality of undetermined coefficients, because $\cos\beta y$ equals 1 and $\mathrm{sh}\beta x$, $\beta x\mathrm{sh}\beta x$, $\beta x\mathrm{ch}\beta x$ are all zero when n equals zero. Corner points O, A, B and C are all the end corner points of supported edges.

Fig. 3 Plates with two opposite edges supported and two edges free

2.4 Rectangular plates with two adjacent edges supported and two adjacent edges free (Fig.4)

$$w_1 = \sum_{m=1,3,5}^{\infty} (A_m \mathrm{sh}\gamma y + B_m \mathrm{ch}\gamma y + C_m \gamma y \mathrm{sh}\gamma y + D_m \gamma y \mathrm{ch}\gamma y)\sin\gamma x + \sum_{n=1,3,5}^{\infty} (E_n \mathrm{sh}\lambda x + F_n \mathrm{ch}\lambda x + G_n \lambda x \mathrm{sh}\lambda x + H_n \lambda x \mathrm{ch}\lambda x)\sin\lambda y + \Delta_O + \frac{(\Delta_A - \Delta_O)y}{b} + \frac{(\Delta_B - \Delta_O)x}{a} \quad (6)$$

where $\gamma = \dfrac{m\pi}{2a}$; $\lambda = \dfrac{n\pi}{2b}$; the polynomial, whose coefficients are made of Δ_O, Δ_A and Δ_B, reflects the deformation characteristic brought on by the displacements of the end corner points of supported edges.

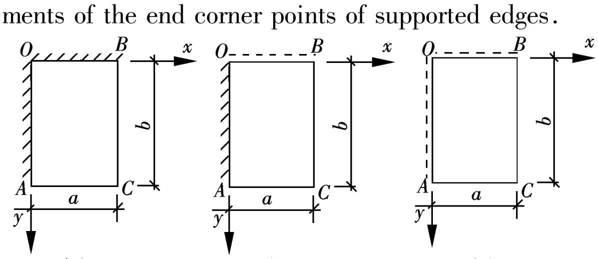

Fig. 4 Plates with two adjacent edges supported and two edges free

2.5 Rectangular plate with one edge simply supported and one pillar support corner point (Fig.5)

$$w_1 = \sum_{m=1,3,5}^{\infty}(A_m \mathrm{sh}\gamma y + B_m \mathrm{ch}\gamma y + C_m \gamma y \mathrm{sh}\gamma y + D_m \gamma y \mathrm{ch}\gamma y)\sin\gamma x + \sum_{n=1,3,5}^{\infty}(E_n \mathrm{sh}\lambda x + F_n \mathrm{ch}\lambda x + G_n \lambda x \mathrm{sh}\lambda x + H_n \lambda x \mathrm{ch}\lambda x)\sin\lambda y + \Delta_O + \frac{(\Delta_A - \Delta_O)y}{b} + \left(\Delta_B - \Delta_O - \sum_{m=1,3,5} B_m \sin\frac{m\pi}{2}\right)\frac{x}{a} \quad (7)$$

where $\gamma = \dfrac{m\pi}{2a}$; $\lambda = \dfrac{n\pi}{2b}$; the polynomial, whose coefficients are made of Δ_O and Δ_A, reflects deformation characteristic brought on by the displacements of the end corner point of supported edge. The corner point B is a pillar support corner point, and its displacement is satisfied by itself.

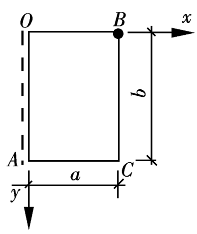

Fig.5 Plate with one edge simply supported and one pillar support corner point

2.6 Rectangular plate with three pillar support corner points (Fig.6)

$$w_1 = \sum_{m=1,3,5}^{\infty}(A_m \mathrm{sh}\gamma y + B_m \mathrm{ch}\gamma y + C_m \gamma y \mathrm{sh}\gamma y + D_m \gamma y \mathrm{ch}\gamma y)\sin\gamma x + \sum_{n=1,3,5}^{\infty}(E_n \mathrm{sh}\lambda x + F_n \mathrm{ch}\lambda x + G_n \lambda x \mathrm{sh}\lambda x + H_n \lambda x \mathrm{ch}\lambda x)\sin\lambda y + \Delta_O + \left(\Delta_B - \Delta_O - \sum_{m=1,3,5} B_m \sin\frac{m\pi}{2}\right)\frac{x}{a} + \left(\Delta_A - \Delta_O - \sum_{n=1,3,5} F_n \sin\frac{n\pi}{2}\right)\frac{y}{b} \quad (8)$$

where $\gamma = \dfrac{m\pi}{2a}$; $\lambda = \dfrac{n\pi}{2b}$; the corner points O, A and B are the pillar support corner points and their displacements are satisfied by themself.

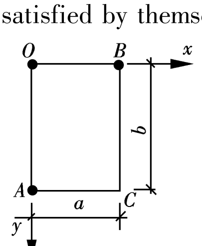

Fig.6 Plate with three pillar support corner points

2.7 Rectangular plate with one clamped edge (Fig.7)

$$w_1 = \sum_{m=1,3,5}^{\infty}(A_m \mathrm{sh}\gamma y + B_m \mathrm{ch}\gamma y + C_m \gamma y \mathrm{sh}\gamma y + D_m \gamma y \mathrm{ch}\gamma y)\sin\gamma x + E_0 x + F_0 + G_0 x^2 + H_0 x^3 + \sum_{n=1,2}^{\infty}(E_n \mathrm{sh}\beta x + F_n \mathrm{ch}\beta x + G_n \beta x \mathrm{sh}\beta x + H_n \beta x \mathrm{ch}\beta x)\cos\beta y + \Delta_O + \frac{(\Delta_A - \Delta_O)y}{b} \quad (9)$$

where $\gamma = \dfrac{m\pi}{2a}$, $\beta = \dfrac{n\pi}{b}$; the polynomial, whose coefficients are made of Δ_O and Δ_A, reflects the deformation characteristic brought on by the displacements of the end corner points of supported edge. The coefficients E_0, F_0, G_0 and H_0 are the E_n, F_n, G_n and H_n respectively when n equals zero.

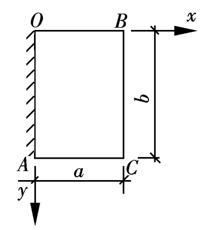

Fig.7 Plate with one clamped edge

3 Particular Solution of Rectangular Plate Bending

Particular solution w_2 is mainly determined by loads. Particular solution is necessary to satisfy Eq.(1) when the plate is subjected to loads $q(x,y)$. Particular solution is also necessary to satisfy the force condition of free corner point when free corner point is subjected to a concentrated load, because the $\dfrac{\partial w_1}{\partial x \partial y} = 0$ at the corner point. Particular solution is not necessary, when the plate edge is subjected to loads or the supported edge has displacements. The particular solution value must be zero at the pillar support corner point, because the homogeneous solution itself satisfies the displacement conditions of this corner point.

3.1 Particular solution w_2 of plate subjected to loads on it

3.1.1 Dual trigonometric series w_2

First we expand $q(x,y)$ into a dual trigonometric series which has the same form as the series used in the homogeneous solution w_1 in order to accord with the deformation character brought on by boundary condition.

For example, when the rectangular plate with three edges supported and one edge free (Fig.2) subjected to uniform load $q(x,y) = q_0$, we take the expression of particular solution as

$$w_2 = \sum_{m=1,3,5} \sum_{n=1,3,5} \frac{16q_0}{mn\pi^2 D(\alpha^2 + \lambda^2)} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{2b} \tag{10}$$

Using orthogonality of series, the functions which are not trigonometric series are expanded into trigonometric series in corresponding zone. Then using boundary conditions of four edges, we can get eight linear equations from which we can derive the eight undetermined coefficients. For the rectangular plate with four edges simply supported the eight undetermined coefficients are zero. We have the classical Navier's solution $w = w_2$.

Using the particular solution in the form of series, we can calculate the bending of the plate subjected to arbitrary loads. The results are shown in Refs. [8 – 14]. When the plate is subjected to a concentrated load at a certain point, the bending moment will not converge at this point.

3.1.2 Polynomial w_2 about x and y

The particular solution can adopt polynomial about x and y, and it must satisfy certain boundary conditions. The calculated results were shown in Refs. [2 – 7].

Although there is great difference in the form between the polynomial and series particular solution, the final results are all the same. Using the polynomial particular solution, we can't calculate the bending of plate subjected to a concentrated load or distributed load in part region, because we can't find the polynomial that satisfies the equilibrium differential equation and before-mentioned conditions.

3.2 Particular solution w_2 of the plate subjected to a concentrated load at free corner

When the plates as shown in Fig.4 – Fig.6 are subjected to a unit concentrated load downward at the corner point C, we have

$$R_C = \left[-2D(1-\mu)\frac{\partial^2 w}{\partial x \partial y}\right]_{x=a,\, y=b} = -1 \tag{11}$$

where μ is Poisson's ratio for the plate material.

The expression $\dfrac{\partial^2 w_1}{\partial x \partial y} = 0$ at the free corner point and w_1 satisfies the deflection condition at the pillar support corner point. Hence the particular solution w_2 must satisfy the concentrated force condition at the free corner point and $w_2 = 0$ at the pillar support corner point. The particular solution must also satisfy certain boundary conditions.

These conditions can be satisfied by taking

$$w_2 = \frac{xy}{2(1-\mu)D} \tag{12}$$

For the plate with two adjacent edges simply supported and two adjacent edges free as shown in Fig.4, if the deflection at the simply supported edges is zero, we obtain that eight undetermined coefficients are all zero when substituting Eq. (12) into boundary conditions. So we have

$$w = w_2 = \frac{xy}{2(1-\mu)D} \tag{13}$$

For the plate with one edge simply supported and one pillar support corner point as shown in Fig.5, if the deflection at the simply supported edge is zero, the eight undetermined coefficients are all zero. We have

$$w = \frac{xy}{2(1-\mu)D} + \frac{\Delta_B x}{a} \tag{14}$$

For the plate with three pillar support corner points as shown in Fig.6, eight undetermined coefficients are all zero. We have

$$w = \frac{xy}{2(1-\mu)D} + \Delta_O + \frac{(\Delta_B - \Delta_O)x}{a} + \frac{(\Delta_A - \Delta_O)y}{b} \tag{15}$$

For the plate with one clamped edge as shown in Fig.7, the particular solution is shown in Ref.[9]. The particular solution w_2 can't take the form of series because the concentrated load at free corner point can't be expanded into series.

4 Bending of Plate Subjected to Edge Loads

When the simply supported edge or the free edge is subjected to distributed or partial distributed bending moment or concentrated moment, or the free edge is subjected to distributed or partial distributed shearing force or concentrated force, the bending can be directly calculated by the homogeneous solution and the boundary conditions. The boundary loads should be expanded into corresponding series.

For the rectangular plate with two opposite edges simply supported and two opposite edges free as shown in Fig.3(a), the plate is subjected to the distributed moment M along the simply supported edges ($x = 0$ and $x = a$) and the distributed moment μM along the free edges ($y = 0$ and $y = b$). Substituting Eq.(5) into boundary conditions and suppose $\Delta_O = \Delta_A = \Delta_B$

$= \Delta_C = 0$, we have that undetermined coefficients equal zero except $G_0 = -\dfrac{M}{2D}$ and $E_0 = \dfrac{Ma}{2D}$. So we obtain

$$w = E_0 x + G_0 x^2 = \frac{M}{2D}(ax - x^2) \tag{16}$$

When the plate as shown in Fig.7 is subjected to the distributed moment M along the free edge ($x = a$) and the distributed moment μM along the free edges ($y = 0$ and $y = b$). Substituting Eq.(9) into boundary conditions and suppose $\Delta_O = \Delta_A = 0$, we have that undetermined coefficients equal zero except $G_0 = -\dfrac{M}{2D}$. So we obtain

$$w = G_0 x^2 = -\frac{Mx^2}{2D} \tag{17}$$

5 Bending of Plate with Boundary Displacements

The bending of the plates, which have normal displacements along the simply supported edges or along the clamped edges or have angle of rotation along the clamped edges, can be calculated directly by the homogeneous solution w_1 and the boundary conditions.

For the rectangular plate with four edges simply supported as shown in Fig.1(a), the plate with three edges simply supported and one edge free as shown in Fig.2(b) and the plate with two opposite edges simply supported and two edges free as shown in Fig.3(a), when the plate has displacements at the four corner points and those supported edges are still linear, we have that eight undetermined coefficients are all zero. That is

$$w = \Delta_O + \frac{(\Delta_B - \Delta_O)x}{a} + \frac{(\Delta_A - \Delta_O)y}{b} + \frac{(\Delta_C - \Delta_A - \Delta_B + \Delta_O)xy}{ab} \tag{18}$$

For the plate with two adjacent edges simply supported and two adjacent edges free as shown in Fig.4(c), when the plate has displacements at the end corner points of supported edge and those supported edges are still linear, we have that eight undetermined coefficients are all zero. So

$$w = \Delta_O + \frac{(\Delta_B - \Delta_O)x}{a} + \frac{(\Delta_A - \Delta_O)y}{b} \tag{19}$$

For the plate with one edge simply supported and one pillar support corner point as shown in Fig.5, when the plate has displacements at the end corner points of supported edge and this supported edge is still linear, we have that eight undetermined coefficients are all zero. The deflection w is the same as Eq.(19).

For the plate with one edge clamped as shown in Fig.7, when the plate has displacements at the end corner points of supported edge and unit angle of rotation along the clamped edge, but this supported edge is still linear, we have that eight undetermined coefficients are all zero except $E_0 = 1$. So

$$w = \Delta_O + \frac{(\Delta_A - \Delta_O)y}{b} + x \tag{20}$$

The results mentioned above are all the same as theoretical solutions, and other results are shown in Ref.[15].

6 Generalized Statically Indeterminate Bending

Using superposition principle the generalized statically indeterminate bending can be solved. Replacing the extra pillar support by an unknown supporting force, we can obtain the statically determinate bending basic system. First we calculate the bending solution of the basic system with the original loads and unit force at the corner point respectively. Then using the deflection condition that the displacements equals zero or a known value at the pillar support corner point, we can obtain the concentrated supporting force. So the problem can be solved using superposition principle.

Bendings of these plates, as shown in Fig.8, Fig.9, Fig.10 and Fig.11, are all generalized statically indeterminate bending problems. When the plate as shown in Fig.8 (c) has arbitrary displacements at the four corner points and the supported edges are

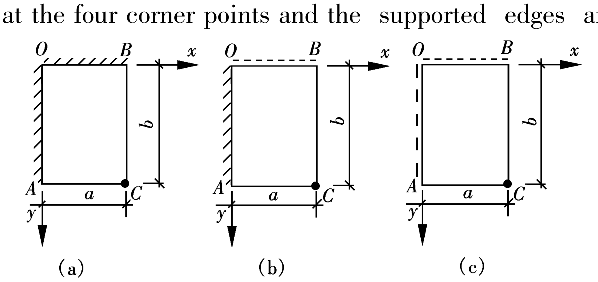

Fig.8 Plates with two adjacent edges supported and one pillar support corner point

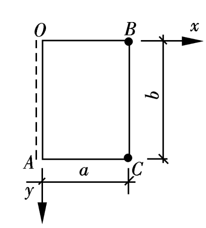

Fig.9 Plate with one edge simply supported and two pillar support corner points

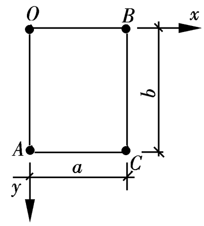

Fig.10 Plate with four pillar support corner points

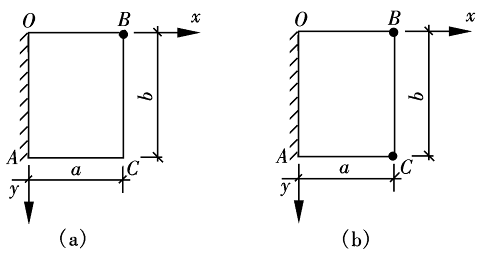

Fig.11 Plates with one clamped edge and one or two pillar support corner points

still linear, using the superposition principle we have

$$w = \Delta_O + \frac{(\Delta_B - \Delta_O)x}{a} + \frac{(\Delta_A - \Delta_O)y}{b} + \frac{(\Delta_C - \Delta_A - \Delta_B + \Delta_O)xy}{ab} \quad (21)$$

When the plate as shown in Fig.9 has arbitrary displacements at four corner points and edge OA is still linear, or when the plate as shown in Fig.10 has arbitrary displacements at four corner points, we have the deflection expression as Eq.(21). The result is the same as theoretical solution.

The calculating results of the generalized statically indeterminate bending of the plates subjected to loads are shown in Refs.[3,5,8,9].

7 Simplified Solution of the Plate with Two Opposite Edges Simply Supported

For the rectangular plates as shown in Fig.1(a), Fig.1(b), Fig.1(c), Fig.2(a), Fig.2(b) and Fig.3(a), the edges $x = 0$ and $x = a$ are all simply supported. When the plates are subjected to loads on them, we can get the undetermined coefficients $E_n = F_n = G_n = H_n = 0$ with the boundary conditions of the edges $x = 0$ and $x = a$. For the plate as shown in Fig.3(a), we also get the coefficients $E_0 = F_0 = G_0 = H_0 = 0$. Eq.(3), Eq.(4) and Eq.(5) can be degenerated into

$$w_1 = \sum_{m=1,2,3} (A_m \mathrm{sh}\alpha y + B_m \mathrm{ch}\alpha y + C_m \alpha y \mathrm{sh}\alpha y + D_m \alpha y \mathrm{ch}\alpha y)\sin\alpha x + \Delta_O + \frac{(\Delta_B - \Delta_O)x}{a} + \frac{(\Delta_A - \Delta_O)y}{b} + \frac{(\Delta_C - \Delta_A - \Delta_B + \Delta_O)xy}{ab} \quad (22)$$

When $\Delta_O = \Delta_A = \Delta_B = \Delta_C = 0$, we can get the classical Levy's homogeneous solution. Because the homogeneous solution only has single trigonometric series, the particular solution can take single trigonometric series as

$$w_2 = \sum_{m=1,2,3} R_m \sin\frac{m\pi x}{a} \quad (23)$$

The classical Levy's solution often take polynomial about x and y as particular solution. The polynomial about x and y is one special form of trigonometric series particular solution, we can get the same series by expanding the polynomial. But the polynomial has the limitation that it can't be used to calculate the bending of plate subjected to discontinuous loads in x or y direction, such as partial distributed loads or concentrated loads. Using the particular solution as shown in Eq.(23), we can calculate the bending of the plate subjected to the loads that are discontinuous in x direction and continuous in y direction. It must adopt the dual trigonometric series particular solution to calculate the discontinuous loads in y direction. The series in y direction should adopt the form as $\sum_{n=1,2,3}\sin\frac{n\pi y}{b}$, $\sum_{n=1,3,5}\sin\frac{n\pi y}{2b}$ and $\sum_{n=0,1,2}\cos\frac{n\pi y}{b}$ respectively, according to different boundary conditions as shown in Fig.1 – Fig.3. Levy's homogeneous solution as shown in Eq.(22) can be applied to the bending of plate subjected to loads or displacements along the edges $y = 0$ and $y = b$. It can't be applied to the bending of plate subjected to arbitrary bending moment along edges $x = 0$ or $x = a$ or the bending of plate with support displacements which are not linear along the edges $x = 0$ or $x = a$, because we can't get the homogeneous solution as shown in Eq.(22) from the boundary conditions of the edges $x = 0$ and $x = a$.

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矩形薄板弹性弯曲统一求解方法

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摘 要 在分析角点求解条件完备性的基础上将矩形板弯曲划分为广义静定问题和广义超静定问题.广义静定弯曲可以由板的平衡微分方程及四边边界条件直接求解,广义超静定弯曲可以由叠加法求解.这种求解方法可以解决各种边界条件下(包括简支边、固定边、自由边、自由角点、支柱角点)的矩形板在任意荷载作用下(包括板面上作用任意法向荷载,板边界上作用任意荷载,板自由角点上作用集中力,板边界及支柱角点发生任意位移)的弯曲.本方法可以将经典的纳维叶解和李维解法有机地统一起来,且收敛速度快,计算精度高.

关键词 弹性薄板弯曲, 矩形板, 统一解法

中图分类号 TU311.4