Journal of Southeast University (English Edition) Sept. 2002 Vol.18 No.3 ISSN 1003—7985

A Result on Multiply Perfect Number

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Abstract: Let n be a positive integer satisfying $n > 1$; $\omega(n)$ denotes the number of distinct prime factors of n; $\sigma(n)$ denotes the sum of the positive divisors of n. If $\sigma(n) = 2n$ then n is said to be a perfect number and if $\sigma(n) = kn(k \geqslant 3)$ then n is said to be a multiply perfect number. In this paper according to Euler theorem and Fermat theorem, we discuss the result of $\sigma(n) = \omega(n)n$ and prove that only $n = 2^3 \cdot 3 \cdot 5,\ 2^5 \cdot 3 \cdot 7,\ 2^5 \cdot 3^3 \cdot 5 \cdot 7$ satisfies $\sigma(n) = \omega(n)n(\omega(n) \geqslant 3)$.

Key words: multiply perfect number, Euler theorem, divisible

Let n be a positive integer satisfying $n > 1$; $\omega(n)$ denotes the number of distinct prime factors of n; $\sigma(n)$ denotes the sum of the positive divisors of n. If $\sigma(n) = kn$ $(k \geqslant 3)$ then n is said to be a multiply perfect number. In this paper we identify all multiply perfect numbers, which are raised in the fashion $\sigma(n) = \omega(n) \cdot n(\omega(n) \geqslant 3)$.

Theorem 1 $n = 2^3 \cdot 3 \cdot 5$ and $n = 2^5 \cdot 3 \cdot 7$ are the only solutions of $\sigma(n) = \omega(n) \cdot n$ with $\omega(n) = 3$.

Theorem 2 $n = 2^5 \cdot 3^3 \cdot 5 \cdot 7$ is the only solution of $\sigma(n) = \omega(n) \cdot n$ with $\omega(n) = 4$.

Theorem 3 There is no positive integer n satisfying $\sigma(n) = \omega(n) \cdot n$ with $\omega(n) \geqslant 5$.

1 Lemmas

To prove the results of this paper, we need the following lemmas.

Lemma 1 Suppose that a, b, k are natural numbers with $a \geqslant 2, b \geqslant 2, k \geqslant 1$ and $b \mid a^k - 1$, $b \nmid a^l - 1$ with $0 < l < k$. If $b \mid a^m - 1$, then $k \mid m$[1].

According to Auler theorem and lemma 1, we have the following specific conclusions.

Lemma 2 ① If $7 \mid 3^{\alpha+1} - 1$, then $13 \mid 3^{\alpha+1} - 1$;

② If $13 \mid 2^{\alpha+1} - 1$, then $7 \mid 2^{\alpha+1} - 1$;

③ If $11 \mid 5^{\alpha+1} - 1$, then $71 \mid 5^{\alpha+1} - 1$;

④ If $11 \mid 2^{\alpha+1} - 1$, then $31 \mid 2^{\alpha+1} - 1$;

⑤ If $7 \mid 5^{\alpha+1} - 1$, then $31 \mid 5^{\alpha+1} - 1$;

⑥ If $2^4 \mid 3^{\alpha+1} - 1$, then $5 \mid 3^{\alpha+1} - 1$.

Lemma 3 Suppose that $p_1, p_2 (p_1 \neq 2)$ are prime numbers and $\alpha + 1$, T, h, k, m are natural numbers with $p_2^T - 1 = p_1^h \cdot k$, $p_1 \nmid k$, $p_1^{h+m} \mid p_2^{\alpha+1} - 1$ and with $0 < t < T$, $p_1 \nmid p_2^t - 1$, then $p_2^{p_1^m} - 1 \mid p_2^{\alpha+1} - 1$.

From $p_1^{h+m} \mid p_2^{\alpha+1} - 1$ we have $p_1 \mid p_2^{\alpha+1} - 1$, from $p_1 \mid p_2^T - 1$, $p_1 \nmid p_2^t - 1 (t < T)$ and lemma 1, we have $T \mid \alpha + 1$. So we can assume $\alpha + 1 = T \cdot p_1^n \cdot k_1$ with $p_1 \nmid k_1$ (n may be 0), then

$$p_2^{T \cdot k_1} = (p_1^h \cdot k + 1)^{k_1} = 1 + p_1^h \cdot k \cdot k_1 + p_1^{2h} \cdot K_1 = 1 + p_1^h \cdot \Delta \text{ with } p_1 \nmid \Delta$$

$$p_2^{T \cdot k_1 \cdot p_1} = (1 + p_1^h \cdot \Delta)^{p_1} = 1 + p_1^{h+1} \Delta + C_{p_1}^2 p_1^{2h} \Delta^2 + \cdots = 1 + p_1^{h+1} \Delta + p_1^{2h+1} K_2 = 1 + p_1^{h+1} \Delta_1$$

Since $p_1 \nmid \Delta$, then $p_1 \nmid \Delta_1$, hence we have $p_2^{T \cdot k_1 \cdot p_1^n} = 1 + p_1^{h+n} \cdot \Delta_n$ with $p_1 \nmid \Delta_n$, that is $p_2^{\alpha+1} - 1 = p_1^{h+n} \cdot \Delta_n$.

From $p_1^{h+m} \mid p_2^{\alpha+1} - 1$ we have $p_1^{h+m} \mid p_1^{h+n} \cdot \Delta_n$ with $p_1 \nmid \Delta_n$, so that $m \leqslant n$ which implies $p_1^m \mid \alpha + 1$. So we have $p_2^{p_1^m} - 1 \mid p_2^{\alpha+1} - 1$.

The following is a specific form of lemma 3.

Lemma 4 ① If $3^2 \mid 2^{\alpha+1} - 1$, then $7 \mid 2^{\alpha+1} - 1$;

Received 2002-03-17.

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② If $3^3 \mid 2^{\alpha+1}-1$, then $73 \mid 2^{\alpha+1}-1$;

③ If $5^2 \mid 2^{\alpha+1}-1$, then $31 \mid 2^{\alpha+1}-1$;

④ If $7^2 \mid 2^{\alpha+1}-1$, then $127 \mid 2^{\alpha+1}-1$;

⑤ If $5^2 \mid 3^{\alpha+1}-1$, then $11 \mid 3^{\alpha+1}-1$;

⑥ If $11^3 \mid 3^{\alpha+1}-1$, then $23 \mid 3^{\alpha+1}-1$;

⑦ If $13^2 \mid 3^{\alpha+1}-1$, then $3^{13}-1 \mid 3^{\alpha+1}-1$;

⑧ If $13^2 \mid 5^{\alpha+1}-1$, then $5^{13}-1 \mid 5^{\alpha+1}-1$.

2 Proof of Theorem 1

Since $\omega(n)=3$, we can assume that $n=p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}$ and p_1, p_2, p_3 are prime numbers with $p_1<p_2<p_3$, $\alpha_i \geqslant 1, i=1,2,3$, from the definition of $\sigma(n)$, we have[2]

$$\sigma(n)=\frac{p_1^{\alpha_1+1}-1}{p_1-1}\cdot\frac{p_2^{\alpha_2+1}-1}{p_2-1}\cdot\frac{p_3^{\alpha_3+1}-1}{p_3-1}=$$
$$\left(p_1^{\alpha_1}+\frac{p_1^{\alpha_1}-1}{p_1-1}\right)\left(p_2^{\alpha_2}+\frac{p_2^{\alpha_2}-1}{p_2-1}\right)\left(p_3^{\alpha_3}+\frac{p_3^{\alpha_3}-1}{p_3-1}\right)<$$
$$p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\left(1+\frac{1}{p_1-1}\right)\left(1+\frac{1}{p_2-1}\right)\left(1+\frac{1}{p_3-1}\right) \tag{1}$$

If $\sigma(n)=\omega(n)\cdot n$ that is $\sigma(n)=3n$, then $p_1=2$, $p_2=3$. If not so, from Eq.(1) we can have $\sigma(n)<3n$ which fails for $\sigma(n)=3n$. So we just discuss the solutions of the following equation:

$$2^{\alpha_1}\cdot 3^{\alpha_2+1}\cdot {p_3}^{\alpha_3}=(2^{\alpha_1+1}-1)\cdot \frac{3^{\alpha_2+1}-1}{2}\,\frac{{p_3}^{\alpha_3+1}-1}{p_3-1} \tag{2}$$

First we draw the conclusion: there is no solution of Eq.(2) unless $2\nmid\frac{3^{\alpha_2+1}-1}{2}$ or $2^2\,\Big\|\,\frac{3^{\alpha_2+1}-1}{2}$ ($p^\alpha \| a$ means $p^\alpha \mid a$ but $p^{\alpha+1}\nmid a$).

We can assume $2^3 \mid \frac{3^{\alpha_2+1}-1}{2}$, it follows from lemma 2 that $5 \mid 3^{\alpha_2+1}-1$ which implies $p_3=5$.

Suppose $5^2 \mid 2^{\alpha_1+1}-1$, from lemma 4 we have $31 \mid 2^{\alpha_1+1}-1$, contradicting Eq.(2) and $p_3=5$; suppose $5^2 \mid \frac{3^{\alpha_2+1}-1}{2}$, from lemma 4 then $11 \mid 3^{\alpha_2+1}-1$, contradicting Eq. (2) and $p_3=5$. The fact $p_3\nmid\frac{p_3^{\alpha_3+1}-1}{p_3-1}$ and these contradictions show $\alpha_3=1$ or 2.

If $\alpha_3=2$, then $\frac{p_3^{\alpha_3+1}-1}{p_3-1}=\frac{5^3-1}{4}=31$, contradicting Eq.(2) which implies $\alpha_3=1$. Hence we have

$$2^{\alpha_1}\cdot 3^{\alpha_2}\cdot 5=(2^{\alpha_1+1}-1)(3^{\alpha_2+1}-1) \tag{3}$$

In Eq.(3), if $3^2 \mid 2^{\alpha_1+1}-1$, from lemma 4 we have $7 \mid 2^{\alpha_1+1}-1$, contradicting Eq.(3) which implies $\alpha_2=1$. So that $\frac{3^{\alpha_2+1}-1}{2}=4$, contradict to the assumption $2^3 \mid \frac{3^{\alpha_2+1}-1}{2}$. Hence in Eq.(2), we have $2^3\nmid\frac{3^{\alpha_2+1}-1}{2}$.

And if $2 \mid \frac{3^{\alpha_2+1}-1}{2}$, then $3^{\alpha_2+1}\equiv 1 \pmod 4$, that is $(-1)^{\alpha_2+1}\equiv 1\pmod 4$, so we have $2 \mid \alpha_2+1$ which implies $3^2-1 \mid 3^{\alpha_2+1}-1$, that is $2^2 \mid \frac{3^{\alpha_2+1}-1}{2}$.

This gives the result of Eq.(2), that is $2\nmid\frac{3^{\alpha_2+1}-1}{2}$ or $2^2\,\Big\|\,\frac{3^{\alpha_2+1}-1}{2}$.

For Eq.(2) we assume $3^2 \mid 2^{\alpha_1+1}-1$, that is we discuss the case $3^2 \mid 2^{\alpha_1+1}-1$.

If $3^2 \mid 2^{\alpha_1+1}-1$, from lemma 4 then $7 \mid 2^{\alpha_1+1}-1$ so that $p_3=7$ in Eq.(2).

Suppose $7^2 \mid 2^{\alpha_1+1}-1$, from lemma 4 then $127 \mid 2^{\alpha_1+1}-1$, contradicting Eq.(2); suppose $7 \mid \frac{3^{\alpha_2+1}-1}{2}$, from lemma 2 then $13 \mid 3^{\alpha_2+1}-1$, contradicting Eq. (2). These contradictions and the fact $p_3\nmid\frac{p_3^{\alpha_3+1}-1}{p_3-1}$ imply $\alpha_3=1$. So from (2), we have

$$2^{\alpha_1-2}\cdot 3^{\alpha_2+1}\cdot 7=(2^{\alpha_1+1}-1)(3^{\alpha_2+1}-1) \tag{4}$$

For Eq. (4) if $3^3 \mid 2^{\alpha_1+1}-1$, from lemma 4 then $73 \mid 2^{\alpha_1+1}-1$, contradict Eq.(4), so that $\alpha_2+1=1$ or 2. And from the fact $\alpha_2\geqslant 1$, we have $\alpha_2=1$. So that we have $2^{\alpha_1-2}\cdot 3^2\cdot 7=(2^{\alpha_1+1}-1)\cdot 8$. This holds only for $\alpha_1=5$, which gives a solution of Eq.(2): $n=2^5\cdot 3\cdot 7$.

From above, there are four cases left to consider for Eq.(2).

Case 1 $2\nmid\frac{3^{\alpha_2+1}-1}{2}$, $3\,\|\,2^{\alpha_1+1}-1$.

From divisibility and Eq.(2), we have

$$2^{\alpha_1}\cdot 3^{\alpha_2}=\frac{p_3^{\alpha_3+1}-1}{p_3-1} \tag{5}$$

and

$$3p_3^{\alpha_3}=(2^{\alpha_1+1}-1)\frac{3^{\alpha_2+1}-1}{2} \tag{6}$$

For Eq.(6), from $3\,\big|\,2^{\alpha_1+1}-1$ we have $\alpha_1+1=2t(t\geqslant 1)$ so that $(2^t+1)(2^t-1)\,\big|\,3p_3^{\alpha_3}$. And also from the fact $(2^t+1,\ 2^t-1)=1$ we have

$$\left.\begin{aligned}2^t-1&=1\\2^t+1&=3p_3^{\beta}\end{aligned}\right\} \tag{7}$$

or

$$\begin{cases}2^t-1=3\\2^t+1=p_3^{\beta}\end{cases}\qquad \beta\leqslant\alpha_3 \tag{8}$$

Eq.(7) holds only for $t=1$, $\beta=0$, so that $\alpha_1=1$. Hence from Eqs.(5) and (6) we have

$$2\cdot 3^{\alpha_2}=\frac{p_3^{\alpha_3+1}-1}{p_3-1} \tag{9}$$

and

$$2p_3^{\alpha_3}=3^{\alpha_2+1}-1 \tag{10}$$

Since $\alpha_3+1\equiv p_3^{\alpha_3}+p_3^{\alpha_3-1}+\cdots+p_3+1=\frac{p_3^{\alpha_3+1}-1}{p_3-1}\equiv 0(\bmod 2)$, then we have $\alpha_3+1=2t_1$. Hence from Eq.(9) we have

$$2\cdot 3^{\alpha_2}=(p_3^{t_1}+1)\frac{p_3^{t_1}-1}{p_3-1} \tag{11}$$

It follows from $(p_3^{t_1}+1,\ p_3^{t_1}-1)=2$ and Eq.(11) that $\left(p_3^{t_1}+1,\ \frac{p_3^{t_1}-1}{p_3-1}\right)=1$.

For Eq.(11) we have $2\,\big|\,p_3^{t_1}+1$, so if $\frac{p_3^{t_1}-1}{p_3-1}\neq 1$, then $p_3^{t_1}+1=2$ and $\frac{p_3^{t_1}-1}{p_3-1}=3^{\alpha_2}$ which implies $t_1=0$, fails for the fact $t_1>0$. So that $\frac{p_3^{t_1}-1}{p_3-1}=1$ which holds only for $t_1=1$, $\alpha_3=1$. So it follows from Eqs.(9) and (10) that $2\cdot 3^{\alpha_2}=p_3+1$ and $2p_3=3^{\alpha_2+1}-1$, so that $4\cdot 3^{\alpha}{}_2=3^{\alpha_2+1}+1$ which holds only for $\alpha_2=0$, fails for $\alpha_2\geqslant 1$.

From Eq.(8), we have $t=2$, $p_3=5$, $\beta=1$, so that $\alpha_1=3$. So it follows from (6) that $5^{\alpha_3-1}=\frac{3^{\alpha_2+1}-1}{2}$. If $\alpha_3-1>1$, then $5^2\,\big|\,3^{\alpha_2+1}-1$, from lemma 4 we have $11\,\big|\,3^{\alpha_2+1}-1$ which fails for $5^{\alpha_3-1}=\frac{3^{\alpha_2+1}-1}{2}$. So that $\alpha_3-1=0$ or 1, that is $\alpha_3=1$ or 2.

If $\alpha_3=1$, then $5^{\alpha_3-1}=\frac{3^{\alpha_2+1}-1}{2}$, so that $\alpha_2+1=1$, which fails for $\alpha_2\geqslant 1$.

If $\alpha_3=2$, then $5^{\alpha_3-1}=\frac{3^{\alpha_2+1}-1}{2}$ have no solution. So there is no solution for Eq.(2) under case 1.

Case 2 $2\nmid\frac{3^{\alpha_2+1}-1}{2}, 3\nmid 2^{\alpha_1+1}-1$.

From divisibility and Eq.(2) we have

$$2^{\alpha_1}\cdot 3^{\alpha_2+1}=\frac{p_3^{\alpha_3+1}-1}{p_3-1} \tag{12}$$

and

$$p_3^{\alpha_3}=(2^{\alpha_1+1}-1)\cdot\frac{3^{\alpha_2+1}-1}{2} \tag{13}$$

From Eq.(13) and the fact $\alpha_1\geqslant 1$, $\alpha_2\geqslant 1$ we have $2^{\alpha_1+1}\equiv 1(\bmod p_3)$ and $3^{\alpha_2+1}\equiv 1(\bmod p_3)$.

It follows (12) that: $2^{\alpha_1+1}\cdot 3^{\alpha_2+1}=2\cdot\frac{p_3^{\alpha_3+1}-1}{p_3-1}=2(p_3^{\alpha_3}+\cdots+p_3+1)$, looking at this equation mod p_3, we obtain $1\equiv 2(\bmod p_3)$, a contradiction. Hence there is no solution under case 2.

Case 3 $2^2\,\big\|\,\frac{3^{\alpha_2+1}-1}{2}$, $3\,\|\,2^{\alpha_1+1}-1$.

It follows from (2) and $2^2\,\big\|\,\frac{3^{\alpha_2+1}-1}{2}$ that $\alpha_1\geqslant 2$. Also from divisibility and (2) we have

$$2^{\alpha_1-2}\cdot 3^{\alpha_2}=\frac{p_3^{\alpha_3+1}-1}{p_3-1} \tag{14}$$

and

$$3p_3^{\alpha_3}=(2^{\alpha_1+1}-1)\frac{3^{\alpha_2+1}-1}{2^3} \tag{15}$$

For Eq.(15), from $3\,\big|\,2^{\alpha_1+1}-1$, that is $(-1)^{\alpha_1+1}\equiv 1(\bmod 3)$ we have $\alpha_1+1=2t$, so that $(2^t+1)\cdot(2^t-1)\,\big|\,3\cdot p_3^{\alpha_3}$, from the fact $(2^t+1,\ 2^t-1)=1$ and $t\geqslant 2$ (since $\alpha_1\geqslant 2$) we have

$$\begin{cases}2^t-1=3\\2^t+1=p_3^{\beta}\end{cases}\qquad \beta\leqslant\alpha_3 \tag{16}$$

Hence from (16) we have $t=2,\beta=1,p_3=5$, so that $\alpha_1=3$, which implies $5^{\alpha_3-1}=\frac{3^{\alpha_2+1}-1}{2^3}$. Suppose $\alpha_3-1>1$, then $5^2\,\big|\,3^{\alpha_2+1}-1$, and from lemma 4, we have $11\,\big|\,3^{\alpha_2+1}-1$, which fails for $5^{\alpha_3-1}=\frac{3^{\alpha_2+1}-1}{2^3}$. So that $\alpha_3-1=0$ or 1, that is $\alpha_3=1$ or 2.

If $\alpha_3=2$, then $5^{\alpha_3-1}=\frac{3^{\alpha_2+1}-1}{2^3}$ holds no solution;

If $\alpha_3=1$, then from $5^{\alpha_3-1}=\frac{3^{\alpha_2+1}-1}{2^3}$ we have $\alpha_2=1$, so that $n=2^3\cdot 3\cdot 5$, which gives anther solution of Eq.(2).

Case 4 $2^2 \left\| \frac{3^{\alpha_2+1}-1}{2}, 3 \nmid 2^{\alpha_1+1}-1 \right.$.

From divisibility and (2) we have

$$2^{\alpha_1-2}\cdot 3^{\alpha_2+1} = \frac{p_3^{\alpha_3+1}-1}{p_3-1} \tag{17}$$

and

$$p_3^{\alpha_3} = (2^{\alpha_1+1}-1)\frac{3^{\alpha_2+1}-1}{2^3} \tag{18}$$

From (18) we have $\frac{3^{\alpha_2+1}-1}{2^3} = p_3^{\beta}(\beta \leqslant \alpha_3)$, and since $4 \mid 3^{\alpha_2+1}-1$ then $\alpha_2+1 = 2t$, so that $(3^t+1)\cdot(3^t-1) = 8\,p_3^{\beta}$. Also from the fact $(3^t+1, 3^t-1) = 2$, we have $3^t-1 = 2$, so that $t = 1, \alpha_2 = 1, \beta = 0$.

Hence from (18) we have $p_3^{\alpha_3} = 2^{\alpha_1+1}-1$, so that $2^{\alpha_1+1} \equiv 1 (\bmod\ p_3)$.

Also from (17), we have: $2^{\alpha_1+1}\cdot 3^2 = 8\cdot\frac{p_3^{\alpha_3+1}-1}{p_3-1} = 8(p_3^{\alpha_3}+\cdots+p_3+1)$, looking at this equation mod p_3, we obtain $8 \equiv 9(\bmod\ p_3)$, a contradiction.

Hence from above we obtain all the solutions of Eq.(2), that is $n = 2^3\cdot 3\cdot 5$ and $n = 2^5\cdot 3\cdot 7$ are the only solutions of $\sigma(n) = \omega(n)\cdot n$ with $\omega(n) = 3$.

3 Proof of Theorem 2 and Theorem 3

As above, we obtain all multiply perfect numbers, which arise in the fashion $\sigma(n) = \omega(n)\cdot n$ with $\omega(n) = 3$. Similarly, we can obtain all multiply perfect numbers in the same fashion with $\omega(n) = 4$.

Since $\omega(n) = 4$, we can assume $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} p_4^{\alpha_4}$, $p_1 < p_2 < p_3 < p_4$ are prime numbers and $\alpha_i \geqslant 1, i = 1,\cdots,4$, then

$$\sigma(n) = \frac{p_1^{\alpha_1+1}-1}{p_1-1}\cdot\frac{p_2^{\alpha_2+1}-1}{p_2-1}\cdot\frac{p_3^{\alpha_3+1}-1}{p_3-1}\cdot\frac{p_4^{\alpha_4+1}-1}{p_4-1} = \left(p_1^{\alpha_1}+\frac{p_1^{\alpha_1}-1}{p_1-1}\right)\left(p_2^{\alpha_2}+\frac{p_2^{\alpha_2}-1}{p_2-1}\right)\cdot\left(p_3^{\alpha_3}+\frac{p_3^{\alpha_3}-1}{p_3-1}\right)\left(p_4^{\alpha_4}+\frac{p_4^{\alpha_4}-1}{p_4-1}\right) < p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} p_4^{\alpha_4}\left(1+\frac{1}{p_1-1}\right)\left(1+\frac{1}{p_2-1}\right)\left(1+\frac{1}{p_3-1}\right)\left(1+\frac{1}{p_4-1}\right) \tag{19}$$

For the least number of p_1, p_2, p_3, p_4 is $2,3,5,7$, so $\sigma(n) < 5n$.

Since $\sigma(n) = 4n$, from (19) we can easily have $p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7$ or 11 or 13.

Case 1 $p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7$.

From $\sigma(n) = 4n$, we have

$$4\cdot 2^{\alpha_1}\cdot 3^{\alpha_2}\cdot 5^{\alpha_3}\cdot 7^{\alpha_4} = (2^{\alpha_1+1}-1)\cdot\frac{3^{\alpha_2+1}-1}{2}\cdot\frac{5^{\alpha_3+1}-1}{4}\cdot\frac{7^{\alpha_4+1}-1}{6} \tag{20}$$

For Eq.(20), suppose $7 \mid 5^{\alpha_3+1}-1$, from lemma 2 we have $31 \mid 5^{\alpha_3+1}-1$, a contradiction, so $7 \nmid 5^{\alpha_3+1}-1$; similarly $7 \nmid 3^{\alpha_2+1}-1, 7^2 \nmid 2^{\alpha_1+1}-1$, so that $\alpha_4 = 1$.

It follows from (20) that:

$$2^{\alpha_1+2}\cdot 3^{\alpha_2}\cdot 5^{\alpha_3}\cdot 7 = (2^{\alpha_1+1}-1)(3^{\alpha_2+1}-1)(5^{\alpha_3+1}-1) \tag{21}$$

For Eq.(21), from lemma 4 we have $5^2 \nmid 3^{\alpha_2+1}-1, 5^2 \nmid 2^{\alpha_1+1}-1$, so that $\alpha_3 = 1$ or 2.

If $\alpha_3 = 2$, then $\frac{5^{\alpha_3+1}-1}{4} = 31$, contradicting (21), which implies $\alpha_3 = 1$.

Hence from (21), we have

$$2^{\alpha_1-1}\cdot 3^{\alpha_2-1}\cdot 5\cdot 7 = (2^{\alpha_1+1}-1)\cdot(3^{\alpha_2+1}-1) \tag{22}$$

From (22) and lemma 4 we also have $3^3 \nmid 2^{\alpha_1+1}-1$, which implies $\alpha_2 = 1,2$ or 3.

If $\alpha_2 = 1$, then Eq.(22) holds no solution. If $\alpha_2 = 2$, then $3^{\alpha_2+1}-1 = 2\times 13$, contradicting (22). If $\alpha_2 = 3$, then $\alpha_1 = 5$, so that $n = 2^5\cdot 3^3\cdot 5\cdot 7$. This gives a solution of Eq.(20).

case 2 $p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 11$.

From $\sigma(n) = 4n$ we have

$$4\cdot 2^{\alpha_1}\cdot 3^{\alpha_2}\cdot 5^{\alpha_3}\cdot 11^{\alpha_4} = (2^{\alpha_1+1}-1)\cdot\frac{3^{\alpha_2+1}-1}{2}\cdot\frac{5^{\alpha_3+1}-1}{4}\cdot\frac{11^{\alpha_4+1}-1}{10} \tag{23}$$

From lemma 2 and (23) we have $11 \nmid 2^{\alpha_1+1}-1$, $11 \nmid 5^{\alpha_3+1}-1$. From lemma 4 we have $11^3 \nmid 3^{\alpha_2+1}-1$. So that $\alpha_4 = 1$ or 2.

Also from (23) and the fact $\alpha_4 \geqslant 1$, we have $11 \mid 3^{\alpha_2+1}-1$, so that $\alpha_2+1 = 5t$, which implies $3^5-1 \mid 3^{\alpha_2+1}-1$, so that $11^2 \mid 3^{\alpha_2+1}-1$, that is $\alpha_4 = 2$.

If $\alpha_4 = 2$, then $11^{\alpha_4+1}-1 = 11^3-1 = 70\times 19$, contradicting (23). So Eq.(23) holds no solution.

Case 3 $p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 13$.

From $\sigma(n) = 4n$, we have

$$4\cdot 2^{\alpha_1}\cdot 3^{\alpha_2}\cdot 5^{\alpha_3}\cdot 13^{\alpha_4} = (2^{\alpha_1+1}-1)\cdot\frac{3^{\alpha_2+1}-1}{2}\cdot\frac{5^{\alpha_3+1}-1}{4}\cdot\frac{13^{\alpha_4+1}-1}{12} \tag{24}$$

From (24) and lemma 2, we have $13 \nmid 2^{\alpha_1+1} - 1$.

Suppose $13^2 \mid 3^{\alpha_2+1} - 1$, from lemma 4 we have $3^{13} - 1 \mid 3^{\alpha_2+1} - 1$, so that $3^{13} - 1$ is a divisor of $3^{\alpha_2+1} - 1$. But from $3^{13} \equiv -1 \pmod 4$, we have $2 \nmid \frac{3^{13}-1}{2}$; from $3^{13} \equiv 3 \pmod 5$, we have $5 \nmid \frac{3^{13}-1}{2}$; from $3^{13} \equiv 3 \pmod{13}$, we have $13 \nmid \frac{3^{13}-1}{2}$ and $3 \nmid \frac{3^{13}-1}{2}$, contradicting (24), so that $13^2 \nmid 3^{\alpha_2+1} - 1$. Similarly, we have $13^2 \nmid 5^{\alpha_3+1} - 1$. Hence $\alpha_4 = 1$ or 2.

If $\alpha_4 = 1$, then $7 \mid 13^{\alpha_4+1} - 1$, contradicting (24). If $\alpha_4 = 2$, then $61 \mid 13^{\alpha_4+1} - 1$, contradicting (24), so Eq. (24) holds no solution. This completes the proof of theorem 2.

For the case $\omega(n) \geqslant 5$, we have the following result.

If $\omega(n) = 5$, then we can assume $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_5^{\alpha_5}$, so that

$$\sigma(n) < p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_5^{\alpha_5} \cdot \left(1 + \frac{1}{p_1 - 1}\right) \cdot \left(1 + \frac{1}{p_2 - 1}\right) \cdots \left(1 + \frac{1}{p_5 - 1}\right)$$

Since the least number of p_1, p_2, p_3, p_4, p_5 is 2, 3, 5, 7, 11, then we have $\left(1 + \frac{1}{p_1 - 1}\right)\left(1 + \frac{1}{p_2 - 1}\right) \cdots \left(1 + \frac{1}{p_5 - 1}\right) < 5$, so that $\sigma(n) < 5n$. Hence when $\omega(n) = 5$, for every natural number n we have $\sigma(n) < \omega(n) \cdot n$.

Suppose for $\omega(n) = k\,(k \geqslant 5)$ the result is right, that is for every natural number n we have $\sigma(n) < \omega(n) \cdot n$. We consider the case $\omega(n) = k + 1$.

Since $\omega(n) = k + 1$, we can assume $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k} \cdot p_{k+1}^{\alpha_{k+1}} = m \cdot p_{k+1}^{\alpha_{k+1}}$, so that

$$\sigma(n) = \sigma(m)\sigma(p_{k+1}^{\alpha_{k+1}}) < \omega(m) \cdot m \cdot \frac{p_{k+1}^{\alpha_{k+1}+1}}{p_{k+1} - 1} = k \cdot n \cdot \frac{p_{k+1}}{p_{k+1} - 1}$$

From $p_{k+1} > k + 1$, we have $\frac{p_{k+1}}{p_{k+1} - 1} < \frac{k+1}{k}$, which implies $\sigma(n) < (k + 1) \cdot n$. This completes the proof of theorem 3.

According to theorems 1, 2, and 3, we have obtained all the solutions of $\sigma(n) = \omega(n) \cdot n\,(\omega(n) \geqslant 3)$. That is only $n = 2^3 \cdot 3 \cdot 5$, $2^5 \cdot 3 \cdot 7$, $2^5 \cdot 3^3 \cdot 5 \cdot 7$ satisfies $\sigma(n) = \omega(n) \cdot n\,(\omega(n) \geqslant 3)$.

References

[1] Birkhoff G D, Vangiver H S. On the integral divisors of $a^n - b^n$[J]. *Ann Math*, 1904, **5**(2): 173 - 180.

[2] Benito Franqui, Mariano Garcia. Some new multiply perfect numbers[J]. *Amer Math Monthly*, 1953, **60**(3): 459 - 462.

关于多重完全数的一个结论

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摘　要　设 n 为大于 1 的正整数,$\omega(n)$表示 n 的不同素因子的个数,$\sigma(n)$为 n 的所有正因子之和. 若 $\sigma(n) = 2n$,则称 n 为完全数. 若 $\sigma(n) = kn\,(k \geqslant 3)$,则称 n 为多重完全数. 本文以欧拉定理及费尔马定理为基础讨论了一种特定条件下的多重完全数问题,即满足 $\sigma(n) = \omega(n) \cdot n\,(\omega(n) \geqslant 3)$的解的情况,得到了 $\sigma(n) = \omega(n) \cdot n\,(\omega(n) \geqslant 3)$的全部解为 $n = 2^3 \cdot 3 \cdot 5, 2^5 \cdot 3 \cdot 7, 2^5 \cdot 3^3 \cdot 5 \cdot 7$.

关键词　多重完全数, 欧拉定理, 整除

中图分类号　O516