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On the Julia Sets of Permutable Transcendental Entire Functions

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Abstract: Let f and g be two permutable transcendental entire functions. In this paper, we first prove that $J(f\circ g) = J(f^n\circ g^m)$ for any positive integers n and m. Then we prove that the function $h(p(z)) + az \notin B$, where $h(z)$ is any transcendental entire function with $h'(z) = 0$ having infinitely many solutions, $p(z)$ is a polynomial with deg $p \geqslant 2$ and $a(\neq 0) \in \mathbf{C}$.

Key words: singular value, Fatou component, permutable transcendental entire function

1 Introduction and Main Results

Let $f(z)$ be a transcendental entire function, and denote by f^n, $n\in\mathbf{N}$, the n-th iterate of f. The set of normality, $F(f)$, is defined to be the set of points, $z\in\mathbf{C}$, such that the sequence $\{f^n\}$ is normal in some neighbourhood of z, and $J = J(f) = \mathbf{C} - F(f)$. $F(f)$ and $J(f)$ are called the Fatou set and Julia set of f, respectively. Clearly $F(f)$ is open, while by the results of Fatou and Julia, $J(f)$ is a nonempty perfect set which coincides with $\mathbf{C}$, or is nowhere dense in $\mathbf{C}$. For the basic results in the dynamical system theory of transcendental functions we refer to Refs.[1 – 3].

Let f be a transcendental entire function, $a\in\mathbf{C}$. If there exists a polynomial $p(z)$ and a nonconstant entire function $h(z)$ such that $f(z) = p(z)e^{h(z)} + a$, then we call a to be a big Picard exceptional value of f, and we denote the set of all such values by $\mathrm{PV}^*(f)$. Furthermore, if $p(z) = (z-a)^k$ for some integer $k\geqslant 0$, then a is said to be a Fatou exceptional value of f. In particular, if $k = 0$, then a is a Picard exceptional value of f. We denote the Fatou exceptional values and the Picard exceptional values of f by $\mathrm{FV}(f)$ and $\mathrm{PV}(f)$, respectively. By Picard theorem, each of the three sets above contains at most one point. Obviously, $\mathrm{PV}(f)\subset\mathrm{FV}(f)\subset\mathrm{PV}^*(f)$.

For two permutable transcendental entire functions f and g, we have $\mathrm{PV}(f\circ g) = \mathrm{PV}(f)\cup\mathrm{PV}(g)$. In fact, if $\mathrm{PV}(f\circ g)$ contains a point x and $x\notin\mathrm{PV}(f)$, then there exists a point z_0 such that $f(z_0) = x$. Note that $f\circ g(z)\neq x$ for any $z\in\mathbf{C}$, thus $g(z)\neq z_0$, i.e., $z_0\in\mathrm{PV}(g)\subset\mathrm{PV}^*(g)$. Since $f(g) = g(f)$, from $x\in\mathrm{PV}(g\circ f)$ we get that $x\in\mathrm{PV}^*(g)$. Thus x, $z_0\in\mathrm{PV}^*(g)$. Since $\mathrm{PV}^*(g)$ contains at most one element, we have $z_0 = x$, and $x\in\mathrm{PV}(g)$. Thus $\mathrm{PV}(f\circ g)\subset\mathrm{PV}(f)\cup\mathrm{PV}(g)$. The converse is obvious.

Let f and g be two nonconstant meromorphic functions. If $f(g) = g(f)$, then we call f and g permutable.

Fatou[4] proved the following results.

Theorem 1 For two given rational functions R_1 and R_2, if they are permutable, then $F(R_1) = F(R_2)$.

The following question is natural (see Ref.[5]).

Question 1 For two given permutable transcendental entire functions f and g, does it follow that $F(f) = F(g)$?

In some special cases, this question was affirmatively solved.

Theorem 2 Let f and g be two permutable transcendental entire functions of finite order. Suppose that

$$f(z) = p_0(z) + p_1(z)e^{q_1(z)}$$

and

$$f(z) = p_0(z) + p_1(z)e^{q_1(z)} + p_2(z)e^{q_2(z)}$$

here $q_i(z)(i = 1, 2)$ and $p_i(z)(i = 0,1,2)$ are polynomials, $p_0(z)$ is a non-constant polynomial. Then $J(f) = J(g)$.

Theorem 3 If f and g are two permutable transcendental entire functions, and there exists a non-constant polynomial $\Phi(x,y)$ in both x and y such

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that $\Phi(f(z), g(z)) \equiv 0$, then $J(f) = J(g)$.

Many other authors studied the dynamical properties of two permutable transcendental entire functions, see Refs. [5 – 9]. Firstly, we prove the following theorems.

Theorem 4 Let f and g be two permutable transcendental entire functions. Then

$$J(f \circ g) = J(f^n \circ g^m) \qquad \forall\, m, n \geqslant 1$$

A point a is called a singular value if it is either a critical value or an asymptotic value. We denote the set of all finite singular values of f by $\mathrm{sing}(f^{-1})$. If the set $\mathrm{sing}(f^{-1})$ is bounded, then we say f is of bounded type, in particular, if the set $\mathrm{sing}(f^{-1})$ is finite, then f is called to be of finite type. We denote them by $f \in B$ and $f \in S$, respectively[1-3].

If a transcendental entire function $f \in B$, then f has many fine dynamical properties[1-3,10]. For any given transcendental entire function f, to determine whether $f \in B$ or not is a difficult problem, and there exists no useful way to do so. But in what follows, we can prove the following results.

Theorem 5 Let $h(z)$ be a transcendental entire function with $h'(z) = 0$ having infinitely many solutions, $p(z)$ be a polynomial with $\deg p \geqslant 2$, and $a(\neq 0) \in \mathbf{C}$. If $f(z) = h(p(z)) + az$, then $f(z) \notin B$.

Theorem 6 Let f be a transcendental entire function. If there exists a sequence $\{z_n\}_{n=1}^{\infty}$ and a positive number $M > 0$ such that $\lim\limits_{n\to\infty} |z_n| = \infty$; $|f'(z_n)| \leqslant M$; $|f^n(z_n)| \leqslant M$.

Then for any non-constant polynomial $q(z)$, we have $f(z) + q(z) \notin B$.

2 Some Lemmas

Lemma 1[11] Let f be a transcendental entire function. Then

$$f^{-1}(F(f)) = F(f) = f(F(f)) \cup \{\mathrm{PV}(f) \cap F(f)\}$$
$$f^{-1}(J(f)) = J(f) = f(J(f)) \cup \{\mathrm{PV}(f) \cap J(f)\}$$

Lemma 2[5] Let f and g be two permutable transcendental entire functions. Then $g(J(f)) \subset J(f)$ and $f(J(g)) \subset J(g)$.

Lemma 3 Let f_1 and f_2 be two permutable transcendental entire functions. Then

$$F(f_1 \circ f_2) \subset F(f_1) \cap F(f_2) \tag{1}$$

Proof Since f_1 and f_2 are permutable, we have

$$f_1 \circ f_1(f_2) = f_1(f_2) \circ f_1$$
$$f_2 \circ f_1(f_2) = f_1(f_2) \circ f_2$$

It follows from lemma 2 that

$$f_1(J(f_1(f_2))),\ f_2(J(f_1(f_2))) \subset J(f_1(f_2)) \tag{2}$$

This and lemma 1 imply that

$$J(f_1(f_2)) = f_1 \circ f_2(J(f_1(f_2))) \cup \{\mathrm{PV}(f_1(f_2)) \cap J(f_1(f_2))\} = f_2 \circ f_1(J(f_1(f_2))) \cup \{\mathrm{PV}(f_1(f_2)) \cap J(f_1(f_2))\} \subset f_2(J(f_1(f_2))) \cup \{\mathrm{PV}(f_1(f_2)) \cap J(f_1(f_2))\} \subset J(f_1(f_2))$$

Thus

$$f_2(J(f_1(f_2))) \cup \{\mathrm{PV}(f_1(f_2)) \cap J(f_1(f_2))\} = J(f_1(f_2)) \tag{3}$$

Similarly we have

$$f_1(J(f_1(f_2))) \cup \{\mathrm{PV}(f_1(f_2)) \cap J(f_1(f_2))\} = J(f_1(f_2)) \tag{4}$$

Next we shall prove

$$f_2^{-1}(J(f_1(f_2))) = J(f_1(f_2)) \tag{5}$$

and

$$f_1^{-1}(J(f_1(f_2))) = J(f_1(f_2)) \tag{6}$$

In fact, for any $a \in f_2^{-1}(J(f_1(f_2)))$, i.e., $f_2(a) \in J(f_1(f_2))$, by (2) we deduce that $f_1(f_2(a)) \in J(f_1(f_2))$. Applying lemma 1 to the function $f_1(f_2)$ we obtain $a \in J(f_1(f_2))$. Hence

$$f_2^{-1}(J(f_1(f_2))) \subset J(f_1(f_2))$$

The converse follows from (3). Thus (5) holds. Eq.(6) can be similarly proved. It follows from (3) – (6) that

$$f_2^{-1}(F(f_1(f_2))) = F(f_1(f_2)) = f_2(F(f_1(f_2))) \cup \{\mathrm{PV}(f_2) \cap F(f_1(f_2))\} \tag{7}$$

In fact, if $b \in F(f_1(f_2)) \setminus \mathrm{PV}(f_2)$, then there exists $c \in \mathbf{C}$ such that $f_2(c) = b$. From (3) we see that $c \in F(f_1(f_2))$, and so $b \in f_2(F(f_1(f_2)))$. Thus

$$F(f_1(f_2)) \subset f_2(F(f_1(f_2))) \cup \{\mathrm{PV}(f_2) \cap F(f_1(f_2))\}$$

All other relations can be similarly proved.

Similarly we have

$$f_1^{-1}(F(f_1(f_2))) = F(f_1(f_2)) = f_1(F(f_1(f_2))) \cup \{\mathrm{PV}(f_1) \cap F(f_1(f_2))\} \tag{8}$$

It follows from (7) and (8) that, for any positive integer k,

$$F(f_1(f_2)) = f_2^k(F(f_1(f_2))) \cup \{\textstyle\bigcup_{j=0}^{k} f_2^j(\mathrm{PV}(f_2)) \cap F(f_1(f_2))\} \tag{9}$$

and

$$F(f_1(f_2)) = f_1^k(F(f_1(f_2))) \cup \{\textstyle\bigcup_{j=0}^{k} f_1^j(\mathrm{PV}(f_2)) \cap F(f_1(f_2))\} \tag{10}$$

In fact, to prove (9), we only need to prove that

$$F(f_1(f_2)) \subset f_2^k(F(f_1(f_2))) \cup$$

$$\{\bigcup_{j=0}^{k} f_2^j(\mathrm{PV}(f_2)) \cap F(f_1(f_2))\} \quad (11)$$

The converse follows from (7). Let

$$a \in F(f_1(f_2)) \setminus \bigcup_{j=0}^{k} f_2^j(\mathrm{PV}(f_2))$$

Then by (7), $a \in f_2(F(f_1(f_2)))$. Thus there exists a point $y_1 \in F(f_1(f_2))$ such that $a = f_2(y_1)$. Note that

$$y_1 \in F(f_1(f_2)) \setminus \bigcup_{j=0}^{k-1} f_2^j(F(f_1(f_2))$$

there exists a point $y_2 \in F(f_1(f_2))$ such that $y_1 = f_2(y_2)$, and so, $a = f_2^2(y_2)$. By induction, there exists a point $y_k \in F(f_1(f_2))$ such that $a = f_2^k(F(f_1(f_2)))$, and so, (11) holds. This proves (9). The proof of (10) is the same.

Combining (9), (10) and Montel's theorem, $\{f_2^k\}$ and $\{f_1^k\}$ are normal in $F(f_1(f_2))$. We thus get (1).

Lemma 4[12] Let f be a transcendental entire function, $n \geqslant 1$. Then we have $F(f) = F(f^n)$.

Lemma 5[13] Let f be a transcendental meromorphic function and $f \in B$. Then, for any $n \geqslant 1$, there exists $K_n > 0$ such that, if $|z| > K_n$ and $|f^n(z)| > K_n$,

$$|(f^n)'(z)| \geqslant \frac{|f^n(z)|\log|f^n(z)|}{16\pi|z|}$$

3 Proof of Theorem 4

For two given positive integers n and m, we shall prove that

$$F(f \circ g) = F(f^n \circ g^m) \quad (12)$$

Let $t > \max\{n, m\}$. From lemma 4 we get

$$F(f \circ g) = F((f \circ g)^t) \quad (13)$$

Now by $f \circ g = g \circ f$, we have

$$(f \circ g)^t = (f^{t-n} \circ g^{t-m}) \circ (f^n \circ g^m) = (f^n \circ g^m) \circ (f^{t-n} \circ g^{t-m})$$

Applying lemma 3 to $f_1 = f^n \circ g^m$ and $f_2 = f^{t-n} \circ g^{t-m}$ we get

$$F((f \circ g)^t) \subset F(f^n \circ g^m) \quad (14)$$

Similarly, we have

$$f^n \circ g^m = (f \circ g) \circ (f^{n-1} \circ g^{m-1}) = (f^{n-1} \circ g^{m-1}) \circ (f \circ g)$$

Applying lemma 3 to $f_1 = f \circ g$ and $f_2 = f^{n-1} \circ g^{m-1}$ we get

$$F(f^n \circ g^m) \subset F(f \circ g)$$

Combining this, (13) and (14), we get (12). The proof is complete.

4 Proofs of Theorems 5 and 6

Proof of Theorem 5 Suppose that $f(z) \in B$, and let $n = \deg p$. Let $\{w_k\}_{k=1}^{\infty}$ be the infinitely many solutions of $h'(w) = 0$, according to the assumption of the theorem, such that $0 \leqslant |w_1| \leqslant |w_2| \leqslant \cdots \leqslant |w_k| \leqslant \cdots \to \infty$ as $k \to \infty$. If we choose R suffciently large, then there exist two branches $a_1(w)$ and $a_2(w)$ of the inverse function p^{-1} of p which are defined for $|w| > R$ and $|\arg w| < \pi$ satisfying the property that

$$a_1(w) \sim \sigma w^{\frac{1}{n}} \text{ and } a_2(w) \sim \sigma e^{\frac{2\pi i}{n}} w^{\frac{1}{n}}$$

as $|w| \to \infty$, here σ is a constant and $w^{\frac{1}{n}}$ denotes the principal branch of the n-th root.

Now for any $k \geqslant 1$, we choose two roots z_k^0, z_k^1 of $p(z) = w_k$ such that

$$z_k^0 \sim \sigma w^{\frac{1}{n}} \text{ and } z_k^1 \sim \sigma e^{\frac{2\pi i}{n}} w^{\frac{1}{n}} \quad (15)$$

Then

$$z_k^0 \to \infty \text{ and } z_k^1 \to \infty \text{ as } k \to \infty \quad (16)$$

$$f'(z_k^0) = h'(p(z_k^0))p'(z_k^0) + a = h'(w_k)p'(z_k^0) + a = a \quad (17)$$

$$f'(z_k^1) = h'(p(z_k^1))p'(z_k^1) + a = h'(w_k)p'(z_k^1) + a = a \quad (18)$$

$$f(z_k^0) = h(p(z_k^0)) + az_k^0 = h(w_k) + az_k^0 \sim h(w_k) + a\delta w_k^{\frac{1}{n}} \quad (19)$$

$$f(z_k^1) = h(p(z_k^1)) + az_k^1 = h(w_k) + az_k^1 \sim h(w_k) + a\delta e^{\frac{2\pi i}{n}} w_k^{\frac{1}{n}} \quad (20)$$

$$f(z_k^0) - f(z_k^1) = a\delta w_k^{\frac{1}{n}}(1 - e^{\frac{2\pi i}{n}}) \quad (21)$$

as $k \to \infty$. Two cases are to be considered.

Case 1 $\{f(z_k^0)\}_{k=1}^{\infty}$ is bounded.

Then by (21), we have

$$f(z_k^1) \sim -a\delta w_k^{\frac{1}{n}}(1 - e^{\frac{2\pi i}{n}}) \to \infty \text{ as } k \to \infty \quad (22)$$

Now from (15), (18), (22) and lemma 5, we derive that,

$$|a| = |f'(z_k^1)| \geqslant \frac{|f(z_k^1)| \cdot \log|f(z_k^1)|}{16\pi|z_k^1|} \sim \frac{|a\delta w_k^{\frac{1}{n}}(1 - e^{\frac{2\pi i}{n}})| \cdot \log|a\delta w_k^{\frac{1}{n}}(1 - e^{\frac{2\pi i}{n}})|}{16\pi|z_k^1|}$$

as $k \to \infty$. Then

$$|16\pi a||z_k^1| \sim |16\pi a\delta w_k^{\frac{1}{n}}| \geqslant |a\delta w_k^{\frac{1}{n}}(1 - e^{\frac{2\pi i}{n}})| \cdot \log|a\delta w_k^{\frac{1}{n}}(1 - e^{\frac{2\pi i}{n}})|$$

as $k \to \infty$, a contradiction.

Case 2 $\{f(z_k^0)\}_{k=1}^{\infty}$ is unbounded.

Then there exists a subsequence of $\{f(z_k^0)\}_{k=1}^{\infty}$ which tends to infinite. For the sake of convenience, we still denote the subsequence as $\{f(z_k^0)\}_{k=1}^{\infty}$. Thus by lemma 5 and (17), we have

$$|a| = |f'(z_k^0)| \geqslant \frac{|f(z_k^0)| \cdot \log|f(z_k^0)|}{16\pi|z_k^0|}$$

this and (16) yield

$$|f(z_k^0)| = o(|z_k^0|)$$

as $k \to \infty$, by (15), we know that $|f(z_k^0)| = o(|w_k^{1/n}|)$ as $k \to \infty$. Then from this and (21), we obtain that

$$f(z_k^1) \sim -a\delta w_k^{1/n}(1 - e^{\frac{2\pi i}{n}}) \tag{23}$$

Now by a similar argument as above, we can also deduce a contradiction. Therefore the theorem follows.

Proof of Theorem 6 Suppose that $f(z) + q(z) \in B$. Let $g(z) = f(z) + q(z)$, then $g(z_n) = f(z_n) + q(z_n)$ and $g'(z_n) = f'(z_n) + q'(z_n)$. Thus $|g'(z_n)| \leqslant |f'(z_n)| + |q'(z_n)| \leqslant M + |q'(z_n)|$, $|g(z_n)| \geqslant |q(z_n)| - M$ and $\lim_{n\to\infty}|g(z_n)| = \infty$. So by the above three inequalities and lemma 5, we deduce that, for suffciently large n,

$$M + |q'(z_n)| \geqslant |g'(z_n)| \geqslant \frac{|g(z_n)| \cdot \log|g(z_n)|}{16\pi|z_n|} \geqslant \frac{(|q(z_n)| - M) \cdot \log(|q(z_n)| - M)}{16\pi|z_n|}$$

This is obviously a contradiction. Therefore $f(z) + q(z) \notin B$.

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关于可交换超越整函数的 Julia 集

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摘　要　令 f 和 g 是两个可交换的超越整函数,本文中我们首先证明对任正整数 n 和 m, $J(f\circ g) = J(f^n \circ g^m)$,然后证明函数 $h(p(z)) + az \notin B$,其中 $h(z)$ 是任超越整函数,且 $h'(z) = 0$ 有无穷多个解,$p(z)$ 是一个多项式,且 $\deg p \geqslant 2$, $a(\neq 0) \in \mathbf{C}$.

关键词　奇异值, Fatou 分量, 可交换的超越整函数

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