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On Miranda's Normal Criterion*

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Abstract: In this paper, we study the normality of a family of analytic functions and prove the following theorem. Let $\mathscr{F}$ be a family of analytic functions in a domain D, k be a positive integer and $a(z)$, $a_1(z)$, $a_2(z)$, $\cdots$, $a_k(z)$ be analytic in D such that $a(z) \not\equiv 0$. If $f(z) \neq 0$ and the zeros of $f^{(k)}(z) + a_1(z)f^{(k-1)}(z) + \cdots + a_k(z)f(z) - a(z)$ are of multiplicity at least 2 for each $f \in \mathscr{F}$, then $\mathscr{F}$ is normal in D. This result improves Miranda's normal criterion.

Key words: entire function, analytic function, normality, differential polynomial

In the theory of normal families of meromorphic functions, it is an important subject searching for new normal criterion[1-3]. In this paper we study the normality of a family of analytic functions. In the following we shall use the usual notations and basic knowledge of normal families[4-6].

In Ref.[7], Montel proposed the following conjecture.

Montel's Conjecture Let $\mathscr{F}$ be a family of analytic functions in a domain D. If, for any $f \in \mathscr{F}$, $f(z) \neq 0$, $f'(z) \neq 1$, then $\mathscr{F}$ is normal in D.

Miranda[8] confirmed the conjecture by proving the following theorems.

Theorem 1 Let $\mathscr{F}$ be a family of analytic functions in a domain D, k be a positive integer. If, for any $f \in \mathscr{F}$, $f(z) \neq 0$, $f^{(k)}(z) \neq 1$, then $\mathscr{F}$ is normal in D.

Valiron and Chuang extended theorem 1 by proving theorem 2[9].

Theorem 2 Let $\mathscr{F}$ be a family of analytic functions in a domain D, k be a positive integer, $a(z), a_1(z), a_2(z), \cdots, a_k(z)$ be analytic functions in the domain D such that $a(z) \not\equiv 0$. If, for any $f \in \mathscr{F}$, $f(z) \neq 0$, $f^{(k)}(z) + a_1(z)f^{(k-1)}(z) + a_2(z)f^{(k-2)}(z) + \cdots + a_k(z)f(z) \neq a(z)$, then $\mathscr{F}$ is normal in D.

Hiong and He[10] improved theorem 1 as follows.

Theorem 3 Let $\mathscr{F}$ be a family of analytic functions in a domain D, k be a positive integer. If $f(z) \neq 0$ and the zeros of $f^{(k)}(z) - 1$ are of multiplicity at least 2 for each $f \in \mathscr{F}$, then $\mathscr{F}$ is normal in D.

Drasin also extended and improved theorem 1 by proving theorem 4[6].

Theorem 4 Let $\mathscr{F}$ be a family of analytic functions in a domain D, k be a positive integer and $a_1(z)$, $a_2(z)$, $\cdots$, $a_k(z)$ be analytic in D. If $f(z) \neq 0$ and the zeros of $f^{(k)}(z) + a_1(z)f^{(k-1)}(z) + \cdots + a_k(z)f(z) - 1$ are of multiplicity at least $k+2$ for each $f \in \mathscr{F}$, then $\mathscr{F}$ is normal in D.

In this note we improve above results by proving theorem 5.

Theorem 5 Let $\mathscr{F}$ be a family of analytic functions in a domain D, k be a positive integer and $a(z)$, $a_1(z)$, $a_2(z)$, $\cdots$, $a_k(z)$ be analytic in D such that $a(z) \not\equiv 0$. If $f(z) \neq 0$ and the zeros of $f^{(k)}(z) + a_1(z)f^{(k-1)}(z) + \cdots + a_k(z)f(z) - a(z)$ are of multiplicity at least 2 for each $f \in \mathscr{F}$, then $\mathscr{F}$ is normal in D.

1 Some Lemmas

For the proof of theorem 5, we need the following lemmas.

Lemma 1[11] Let $\mathscr{F}$ possess the property that every function $f \in \mathscr{F}$ has only zeros of multiplicity at least k. If $\mathscr{F}$ is not normal at point z_0, then for $0 \leqslant \alpha < k$, there exists a sequence of functions $f_n \in \mathscr{F}$, a sequence of complex numbers $z_n \to z_0$ and a sequence of positive numbers $\rho_n \to 0$, such that $\rho_n^{-\alpha} f_n(z_n + \rho_n \zeta)$ converges locally uniformly to a non-constant function $g(\zeta)$ on $\mathbf{C}$.

Lemma 2 Let $f(z)$ be an entire function, k be a positive integer, and d be a nonzero finite complex number. If $f(z) \neq 0$ and the zeros of $f^{(k)}(z) - d$ are

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of multiplicity at least 2, then $f(z)$ is a constant.

Proof Suppose that $f(z)$ is a transcendental entire function, then by Milloux's inequality, Nevanlinna's first fundamental theorem[4-6] and the condition of the lemma, we have

$$T(r,f) \leqslant \bar{N}(r,f) + N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f^{(k)}-d}\right) - N\left(r,\frac{1}{f^{(k+1)}}\right) + S(r,f) \leqslant \bar{N}\left(r,\frac{1}{f^{(k)}-d}\right) + S(r,f) \leqslant \frac{1}{2}N\left(r,\frac{1}{f^{(k)}-d}\right) + S(r,f) \leqslant \frac{1}{2}T\left(r,\frac{1}{f^{(k)}-d}\right) + S(r,f) \leqslant \frac{1}{2}T(r,f^{(k)}-d) + S(r,f) \leqslant \frac{1}{2}T(r,f) + S(r,f)$$

Thus we obtain $T(r,f) = S(r,f)$, a contradiction.

Hence $f(z)$ is polynomial. Considering $f(z) \neq 0$, we deduce that $f(z)$ is a constant. Thus lemma 2 is proved.

2 Proof of Theorem 5

We assume that $D = \{z: |z| < 1\}$ is the unit disc. In the following we prove that $\mathscr{F}$ is normal at z_0 for any $z_0 \in D$. Now we consider two cases.

Case 1 $a(z_0) \neq 0$. Suppose that $\mathscr{F}$ is not normal at $z_0 \in D$. Then taking $\alpha = k(<\infty)$ and according to lemma 1, there exists a sequence of complex numbers $z_n \to z_0$, a sequence of functions $f_n(z) \in \mathscr{F}$, a sequence of positive numbers $\rho_n \to 0$ such that

$$g_n(\xi) = \rho_n^{-k} f_n(z_n + \rho_n \xi) \to g(\xi)$$

uniformly on any compact subset of $\mathbf{C}$, where $g(\xi)$ is a non-constant entire function.

Considering $f_n(z) \in \mathscr{F}$ and $f_n \neq 0$, we deduce by the Hurwitz's theorem that $g(\xi) \neq 0$.

If $g^{(k)}(\xi) - a(z_0) \neq 0$, then by lemma 2 we get $g(\xi)$ is a constant. This is impossible. So there exists a point ξ_0 such that

$$g^{(k)}(\xi_0) - a(z_0) = 0 \tag{1}$$

For any $R > 0$, we know that $g_n^{(i)}(\xi)$ are analytic on $D_R = \{\xi: |\xi - \xi_0| < R\}$ for $i = 1,\cdots,k$ and sufficiently large n. It follows by simple computing that

$$g_n^{(k)}(\xi) - a(z_n + \rho_n\xi) = f_n^{(k)}(z_n + \rho_n\xi) - a(z_n + \rho_n\xi) = f_n^{(k)}(z_n + \rho_n\xi) + M(f_n(z_n + \rho_n\xi)) - a(z_n + \rho_n\xi) - M(f_n(z_n + \rho_n\xi)) \tag{2}$$

where $M(f_n(z_n + \rho_n\xi)) = \sum_{j=0}^{k-1} a_{k-j}(z_n + \rho_n\xi) f^{(j)}(z_n + \rho_n\xi) = \sum_{j=0}^{k-1} a_{k-j}(z_n + \rho_n\xi)\rho_n^{-j+k} g_n^{(j)}(\xi)$ converges to zero uniformly on D_R.

Clearly, $L_n(\xi) = g_n^{(k)}(\xi) + M(f_n(z_n + \rho_n\xi)) - a(z_n + \rho_n\xi) = 0$ have solutions with multiplicity at least 2 on D_R. Combining the Hurwitz's theorem and (2), we deduce that $L_n(\xi)$ converges to $g^{(k)}(\xi) - a(z_0)$ and the zeros of $g^{(k)}(\xi) - a(z_0)$ are of multiplicity at least 2 on D_R. Let $R \to \infty$, we get that $g(\xi) \neq 0$, and the zeros of $g^{(k)}(\xi) - a(z_0)$ are of multiplicity at least 2 on $\mathbf{C}$. Thus by lemma 2 we know that $g(\xi)$ is a constant, which is a contradiction. Hence $\mathscr{F}$ is normal at z_0.

Case 2 $a(z_0) = 0$. Without loss of generality we assume that $z_0 = 0$. Hence there exists $r(0 < r < 1)$ such that $a(z) \neq 0$ in $\{0 < |z| \leqslant r\}$. Thus for any $f_n(z) \in \mathscr{F}$, we know $\{f_n(z)\}$ is normal in $C_r = \{z: |z| = r\}$ by former conclusion. Thus there exists a subsequence f_{n_k} such that

$$f_{n_k}(z) \to g(z)$$

uniformly on C_r.

If $g(z) \not\equiv \infty$, then $g(z)$ is analytic on C_r. Hence there exists an integer N and a positive number M such that

$$|f_{n_k}(z)| \leqslant M$$

for all $k \geqslant N, z \in C_r$. By maximum modulus theorem, we have

$$|f_{n_k}(z)| \leqslant M$$

for all $k \geqslant N$, $|z| \leqslant r$. Hence $\{f_{n_k}(z)\}$ is normal in $\{z: |z| \leqslant r\}$ by Montel's criterion[4-6]. Thus there exists a subsequence of $f_{n_k}(z)$ (we also denote them by $f_{n_k}(z)$) such that

$$f_{n_k}(z) \to g(z) \tag{3}$$

uniformly on $\{z: |z| \leqslant r\}$.

If $g(z) \equiv \infty$, then for any positive M there exists an integer N such that

$$|f_{n_k}(z)| \geqslant M$$

for all $k \geqslant N$, $z \in C_r$. Thus

$$|f_{n_k}(z)| \geqslant M > 0$$

for all $k \geqslant N$, $z \in C_r$. Noting that $f_{n_k}(z)$ has no zeros in $\{z: |z| \leqslant r\}$, we know

$$|f_{n_k}(z)| \geqslant M$$

for all $k \geqslant N$, $|z| \leqslant r$ by the minimum modular theorem. This means that

$$f_{n_k}(z) \to \infty \tag{4}$$

uniformly on $\{z: |z| \leqslant r\}$. Thus we deduce from (3) and (4) that $\mathscr{F}$ is normal at $z_0 = 0$. Hence we deduce that $\mathscr{F}$ is normal at z_0, that is $\mathscr{F}$ is normal in D. Thus the proof of theorem 5 is complete.

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关于 Miranda 正规准则

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摘　要　本文研究了解析函数族的正规性,得到了下面的结论:设 $\mathscr{F}$ 是一族在 D 上的解析函数,k 是一个正整数,$a(z)$,$a_1(z)$,$a_2(z)$,$\cdots$,$a_k(z)$都在 D 上解析且 $a(z)\neq 0$,如果 $f(z)\neq 0$ 并且对 $\mathscr{F}$ 中的任一函数 $f(z)$,$f^{(k)}(z)+a_1(z)f^{(k-1)}(z)+\cdots+a_k(z)f(z)-a(z)$的零点都至少是二级或二级以上,则 $\mathscr{F}$ 在 D 上正规.

关键词　整函数,解析函数,正规性,微分多项式

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