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The Star-Extremality of Circulant Graphs

Wu Jianzhuan* Xu Kexiang

(Department of Mathematics, Southeast University, Nanjing 210096)

Abstract: The circular chromatic number and the fractional chromatic number are two generalizations of the ordinary chromatic number of a graph. We say a graph G is star-extremal if its circular chromatic number is equal to its fractional chromatic number. This paper gives an improvement of a theorem. And we show that several classes of circulant graphs are star-extremal.

Key words: circular chromatic number, fractional chromatic number, circulant graph, star-extremal graph

Let k and d be two natural numbers such that $k \geqslant 2d$. A (k,d)-coloring of a graph $G=(V,E)$ is a mapping $c: V \to \{0,1,2,\cdots,k-1\}$, such that, for each edge $uv \in E$, $|c(u)-c(v)|_k \geqslant d$, where $|x|_k = \min\{|x|, k-|x|\}$. Observe that a $(k,1)$-coloring of a graph G is just an ordinary k-coloring of G. We say G is (k,d)-colorable if there exists a (k,d)-coloring of G. The circular chromatic number (also known as the star chromatic number which was first introduced by Vince[1]) $\chi_c(G)=\min\{k/d: G \text{ is } (k,d)\text{-colorable}\}$. For a different but equivalent definition of the circular chromatic number, we refer the readers to Ref.[2]. The circular chromatic number is a natural generalization of the ordinary chromatic number and can be viewed as a refinement of the ordinary chromatic number.

Another generalization of the ordinary chromatic number is the fractional chromatic number of a graph G. A mapping c from the collection Γ of independent sets of a graph G to the interval $[0,1]$ is a fractional coloring if for every vertex x of G, we have $\sum_{S\in\Gamma,\text{s.t.}\, x\in S} c(S)=1$. The value of a fractional coloring c is $\sum_{S\in\Gamma} c(S)$. The fractional chromatic number $\chi_f(G)$ of G is the infimum of the values of fractional colorings of G. For equivalent definitions of the fractional chromatic number, see Ref.[3]. For any graph G, it is well known that[2]

$$\max\left\{\omega(G), \frac{|V(G)|}{\alpha(G)}\right\} \leqslant \chi_f(G) \leqslant \chi_c(G) \leqslant \chi(G),\ \lceil \chi_c(G) \rceil = \chi(G) \tag{1}$$

where $\alpha(G)$ is the independence number of G. A graph G is called star extremal if $\chi_c(G)=\chi_f(G)$, which is the equality holds in the second inequality in formula (1). Gao and Zhu first introduced this notion of star extremality for graphs when they studied the chromatic number and circular chromatic number of the lexicographic product of graphs in Ref.[4]. Let p be a positive integer and let S be a subset of $\{0,1,2,\cdots,p-1\}$ such that $i\in S$ implies $p-i\in S$. For brevity, $p-i$ is denoted by $-i$. The circulant graph $G(p,S)$ has vertices $0,1,2,\cdots,p-1$ and i is adjacent to j if and only if $i-j\in S$, where subtraction is carried out modulo p. It is easy to prove that if a graph G is vertex transitive, then $\chi_f(G)=\frac{|V(G)|}{\alpha(G)}$, where $\alpha(G)$ is its independence number. Since any circulant graph $G=G(p,S)$ is vertex transitive, we have $\chi_f(G)=\frac{p}{\alpha(G)}$. Thus, to prove that $\chi_c(G)=\chi_f(G)$ for a circulant graph $G=G(p,S)$, it is sufficient to prove that $\chi_c(G)=\frac{p}{\alpha(G)}$.

Given a circulant graph $G=G(p,S)$ and an integer t, let $\lambda_t(G)=\min\{|ti|_p: i\in S\}$ and $\lambda(G)=\max\{\lambda_t(G), t=1,2,\cdots\}$, where the multiplications ti are taken modulo p.

Lemma 1[4] Suppose G is a circulant graph, then $\lambda(G)\leqslant\alpha(G)$. Moreover, if $\lambda(G)=\alpha(G)$, then G is star-extremal.

Lemma 2[5] Let $G=G(p,S)$ be a circulant graph. Suppose $S=\{\pm k, \pm(k+1),\cdots,\pm k'\}$ and $p=q(k+k')+r$, where $0\leqslant r\leqslant k+k'-1$. Then

$$\lambda(G)\geqslant \begin{cases} \lambda_q(G)=qk & 0\leqslant r\leqslant k' \\ \lambda_{q+1}(G)=qk+r-k' & k'+1\leqslant r\leqslant k'+k-1 \end{cases}$$

Theorem 1 Suppose $S=\{\pm k, \pm(k+1),\cdots,$

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* Born in 1970, female, lecturer.

$\pm k'\}$ and $G = G(p, S)$. Write $p = q(k + k') + r$, where $0 \leqslant r \leqslant k + k' - 1$. If $k' \geqslant (5/4)k - 1/2$, then $\alpha(G) = \lambda(G) = qk + \max\{0, r - k'\}$ and G is star-extremal.

Proof By lemma 1, it is sufficient to show that $\alpha(G) \leqslant \lambda(G)$. If $q = 0$, then $p = r \leqslant k + k' - 1$. On the other hand, we always assume $k \leqslant k' \leqslant p/2$. This is a contradiction. Thus we may assume $q \geqslant 1$. Suppose to the contrary that $\alpha(G) > \lambda(G) = qk + \max\{0, r - k'\}$. Let I be an independent set of G. We call $\{i, i+1, \cdots, i'\}$ an I-interval if $\{i, i+1, \cdots, i'\} \subseteq I$ and $i - 1, i' + 1 \notin I$. Clearly I is the disjoint union of I-intervals. Choose I among all maximum independent sets such that the number of its I-intervals is minimum. Let $[a, b]$ and $[c, d]$ be two consecutive I-intervals, where "consecutive" means $[b+1, c-1] \cap I$ is empty. From the proof of theorem 2.8 in Ref. [5], we have

1) $b + k' + 1 \leqslant c + k - 1$ and $b - k + 1 \leqslant c - k' - 1$;

2) $|[b+1, c-1]| \leqslant k - 2$;

3) Let $t = |[b+1, c-1]|$, then $2(k - k') + 1 \leqslant t \leqslant k - 2$. And since $k' \geqslant (5/4)k - 1/2$, $k/2 \leqslant t \leqslant k - 2$;

4) If $|[a, d]| \leqslant k$, then $d \geqslant a + k$. Since $c \geqslant b + k' + 1$, we have $|[b+1, c-1]| = c - b - 1 \geqslant k' + 1 \geqslant (5/4)k - (1/2) + 1 > k - 2$, a contradiction. Thus $|[a, d]| \leqslant k$ and $|[a, b]| + |[c, d]| \leqslant k - [2(k' - k) + 1] \leqslant k/2$. This implies that there are at least four I-intervals.

Now choose two I-intervals $[a, b]$ and $[c, d]$ such that $|[a, b]| + |[c, d]|$ is as large as possible. Let $[u, v]$ be the I-interval preceding $[a, b]$ (i.e. $[u, v]$ and $[a, b]$ are consecutive I-intervals) and let $[x, y]$ be the I-interval following $[c, d]$. From the proof of theorem 2.8 in Ref. [5], $[x, y]$ is the unique I-interval included in $[b + k' + 1, c + k - 1]$ and $[u, v]$ is the unique I-interval included in $[c - k' - 1, b - k + 1]$. From the choice of $[a, b]$ and $[c, d]$, we have $|[u, v]| \leqslant |[c, d]|$ and $|[x, y]| \leqslant |[a, b]|$. Hence $|[u, v]| + |[x, y]| \leqslant |[a, b]| + |[c, d]| \leqslant k/2$. Let $I' = (I \cup [b+1, c-1]) - ([u, v] \cup [x, y])$. Clearly I' is independent and $|I'| \geqslant |I|$. This contradicts the assumption that I is a maximum independent set with the number of its I-intervals minimum. Thus $\alpha(G) = \lambda(G) = qk + \max\{0, r - k'\}$ and G is star-extremal.

Theorem 2 Suppose $G = G(p, S)$ is a circulant graph with $S = \{\pm k, \pm(k+1), \cdots, \pm k', p/2\}$, where p is even. Write $p = q(k + k') + r(0 \leqslant r \leqslant k + k' - 1)$. If $k' \geqslant (5/4)k - 1/2$, then

① If q is odd and $0 \leqslant r \leqslant k'$, then $\alpha(G) = \lambda(G) = qk$ and G is star-extremal;

② If q is even and $k' + 1 \leqslant r \leqslant k + k' - 1$, then $\alpha(G) = \lambda(G) = qk + r - k'$ and G is star-extremal.

Proof of ① We only need to show that $\alpha(G) \leqslant \lambda(G)$. By theorem 1, since $0 \leqslant r \leqslant k'$, $\alpha(G) \leqslant qk$. Since $p - k'q = kq + r \geqslant kq$ and $(p/2)q (\mathrm{mod}\ p) = p/2 \geqslant kq$, we have $\lambda_q(G) = \min\{|kq|_p, |(k+1)q|_p, \cdots, |k'q|_p, |(p/2)q|_p\} = kq$. Thus $\alpha(G) \leqslant \lambda(G)$ and G is star-extremal.

The proof of ② is very similar to that of ①, so we omit it.

To prove the following theorem we first give a simple observation.

Lemma 3 Suppose $G = G(p, S)$ is a circulant graph with $S = \{\pm k, \pm(k+1), \cdots, \pm k', p/2\}$, where p is even. If $\alpha(G) = k$, then $p \leqslant 2k + 2k'$.

Proof Suppose $p > 2k + 2k'$, then it is easy to check that $\{0, 1, \cdots, k - 1, k + k' + 1\}$ is an independent set of G. This is a contradiction since $\alpha(G) = k$.

Theorem 3 Suppose $G = G(p, S)$ is a circulant graph with $S = \{\pm k, \pm(k+1), \cdots, \pm k', p/2\}$, where p is even and $k' > k \neq 1$. If $p = 2k' + 2$, $2k' + 4$, $2k' + 2k$ or $2k' + 2k - 2$, then $\alpha(G) = k$ and G is star-extremal.

Proof Clearly $\alpha(G) \geqslant k$. Following we shall show that $\alpha(G) \leqslant k$. Otherwise suppose $\alpha(G) \geqslant k + 1$. Let A be a maximum independent set of G. Since $\alpha(G) \geqslant k + 1$, $|A| \geqslant k + 1$. This implies that A has at least two intervals. Consider one interval $I_1 = \{i, i + 1, \cdots, i'\}$ $(i' = i + r)$ of A. Let M be the set of vertices of G that are not in I_1 and also not adjacent to any vertex of I_1. It is easy to see that M consists of the following four intervals:

$X = \{i' - k + 1, \cdots, i - 1\}$

$Y = \{i' + 1, \cdots, i + k - 1\}$

$Z_1 = \{i' + k' + 1, \cdots, i + p/2 - 1\}$

$Z_2 = \{i' + p/2 + 1, \cdots, i + p - k' - 1\}$

Obviously $A \setminus I_1 \subset M$.

1) When $p = 2k' + 2$, then $|Z_1| = |Z_2| = 0$ and $|X| = |Y|$. Since for any $t \in X$ we have $t + k \in Y$, $|A| \leqslant |I_1| + |X| = k$. A contradiction.

2) When $p = 2k' + 4$, then $|X| = |Y| = k - 1 - r$ and $|Z_1| = |Z_2| = p/2 - k' - 1 - r = 1 - r$. If $r = 1$, then $|Z_1| = |Z_2| = 0$. Thus $|A| \leqslant |I_1| + |X| = k$. A contradiction. If $r = 0$, then $|Z_1| = |Z_2| = 1, i = i'$, $Z_1 = \{i + k' + 1\}$ and Z_2

$= \{i - k' - 1\}$. Since $i + k' + 1$ is adjacent to $i - 1$, $i - 3$ and $i + 1$, and $i - k' - 1$ is adjacent to $i - 1$, $i + 1$ and $i + 3$, we have $|A| \leqslant |I_1| + |X| = k$. A contradiction.

3) When $p = 2k' + 2k - 2$, then $|X| = |Y| = k - 1 - r$ and $|Z_1| = |Z_2| = k - 1 - r - 1$. Consider the following four vertex sets.

$Y = \{i' + 1, \cdots, i + k - 1\}$

$X = \{i' - k + 1, \cdots, i - 1\}$

$D_1 = Z_1 \cup \{i' + k'\} = \{i' + k', \cdots, i + (p/2) - 1\}$

$D_2 = Z_2 \cup \{i - k'\} = \{i' + (p/2) + 1, \cdots, i + p - k'\}$

Obviously $|X| = |Y| = |D_1| = |D_2|$. It is not difficult to check that for $t = 1, 2, \cdots, k - r - 1$, $\{i' - k + t, i' + t, i' + k' + t, i' + (p/2) + t\}$ induces a clique of G. Thus $|A| \leqslant |I_1| + |X| = k$. This is a contradiction.

4) When $p = 2k' + 2k$, then $|X| = |Y| = |Z_1| = |Z_2| = k - 1 - r$. Similar to 3), it is easy to see that for $t = 1, 2, \cdots, k - r - 1$, $\{i' - k + t, i' + t, i' + k' + t, i' + (p/2) + t\}$ induces a clique of G. Hence $|A| \leqslant |I_1| + |X| = k$. This is a contradiction.

Therefore $\alpha(G) = k$. Since $\lambda(G) \geqslant \lambda_1(G) = k$, we have $\chi_c(G) = \chi_f(G)$ by lemma 1. This completes the proof of theorem 3.

Theorem 4 Suppose $G = G(p, S)$ is a circulant graph with $S = \{\pm k, \pm(k + 1), \cdots, \pm k', \pm(p - 1)/2\}$, where p is odd and $k' > k \neq 1$. If $p = 2k' + 1$, $2k' + 3$, $2(k' + k) - 1$ or $2(k' + k) + 1$, then $\alpha(G) = k$ and G is star-extremal.

The proof of theorem 4 is very similar to that of theorem 3, so we omit it here.

Remark Suppose $G = G(p, S)$ is a circulant graph with $S = \{\pm k, \pm(k + 1), \cdots, \pm k', p/2\}$ and p is even. For $2k' < p \leqslant 2k + 2k'$, it is not always the case that $\alpha(G) = k$. For example, when $k = 7$, $k' = 8$, and $p = 2k' + 2k - 4 = 26$ we have $\alpha(G) = 9 > 7$. And when $k = 5$, $k' = 6$, and $p = 2k' + 6 = 18$ it holds $\alpha(G) = 6 > 5$. It seems very difficult to determine the independence number of $G = G(p, S)$ for all values $2k' < p \leqslant 2k + 2k'$.

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循环图的 star-extremal 性质

吴建专　　许克祥

(东南大学数学系,南京 210096)

摘　要　圆色数和分式色数是图的点色数的两个推广.当图的圆色数等于分式色数时,我们称此图是star-extremal.本文给出了一个定理改进,同时给出了几类具有 star-extremal 特征的循环图.

关键词　圆色数,分式色数,循环图,star-extremal 图

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