Method based on fuzzy linguistic scale and FIOWGA operator for decision-making problems

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Abstract: In this paper, we present a fuzzy linguistic scale, which is characterized by triangular fuzzy numbers on [1/9, 9], for the comparison between two alternatives, and introduce a possibility degree formula for comparing triangular fuzzy numbers. We utilize the fuzzy linguistic scale to construct a linguistic preference matrix, and propose a fuzzy induced ordered weighted geometric averaging (FIOWGA) operator to aggregate linguistic preference information. A method based on the fuzzy linguistic scale and FIOWGA operator for decision-making problems is presented. Finally, an illustrative example is given to verify the developed method and to demonstrate its feasibility and effectiveness.

Key words: fuzzy linguistic scale; triangular fuzzy numbers; FIOWGA operator

Decision-making problems generally consist of finding the most desirable alternative(s) from a given alternative set. In the process of decision-making, the decision maker (DM) generally needs to compare a set of decision alternatives with respect to a single criterion, and then to construct a judgement matrix with a scale of 1 to 9[1]. However, many decision-making processes, in the real world, take place in an environment in which the information is not precisely known. The DM may have vague knowledge about the preference degree of one alternative over another, and cannot estimate his/her preference with an exact numerical value, but with an interval number[2] or a triangular fuzzy number[3]. In some situations, which are too complex or too ill-defined to be amenable to description in conventional quantitative expressions, it is more suitable to provide his/her preference by means of a fuzzy linguistic variable. A fuzzy linguistic variable differs from a numerical one in that its value is not a number, but a word or a sentence in a natural or artificial language[4]. After constructing a judgement matrix, the DM then needs to aggregate the preference information contained in the judgement matrix, and to rank the given alternatives. One of the most common methods for aggregating decision information is the ordered weighted averaging (OWA) operator presented by Yager[5]. Some additional families of OWA operators have been introduced in Refs.[6-11]. In the short time since their first appearance, the OWA operators have been used in an astonishingly wide range of applications[12]. Most of these operators, however, can only be used in situations where the input arguments are the exact values, and few of them can be used to aggregate the linguistic preference information[7,8]. In this paper, we develop a new method to aggregate the fuzzy linguistic preference information and then to rank the given alternatives.

To do so, this paper is structured as follows: Section 1 introduces a possibility degree formula for comparing triangular fuzzy numbers. Section 2 develops a fuzzy induced ordered weighted geometric averaging (FIOWGA) operator. Section 3 gives a fuzzy linguistic scale for the comparison between two alternatives. Section 4 presents an approach, based on the linguistic scale and FIOWGA operator, to ranking alternatives. An illustrative example is given to verify the developed method and to demonstrate its feasibility and effectiveness in section 5. Section 6 contains our conclusion.

1 A Possibility Degree Formula for Comparing Triangular Fuzzy Numbers

Let Ψ be a set of triangular fuzzy numbers. $\hat{a} = [a^L, a^M, a^U] \in \Psi$ will be identified by its characteristic function[13]:

$$\mu_{\hat{a}}(x) = \begin{cases} 0 & 0 \leqslant x \leqslant a^L \\ \dfrac{x - a^L}{a^M - a^L} & a^L \leqslant x \leqslant a^M \\ \dfrac{x - a^U}{a^M - a^U} & a^M \leqslant x \leqslant a^U \\ 0 & x \geqslant a^U \end{cases}$$

where $0 < a^L \leqslant a^M \leqslant a^U$, a^L and a^U stand for the lower

Received 2002-06-27.

Foundation items: The National Natural Science Foundation of China (79970093) and the Ph.D. Dissertation Foundation of Southeast University - NARI-Relays Electric Co. Ltd.

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and upper values of the support of $\hat{a}$, respectively, and a^{M} for the modal value.

Definition 1 Let $\hat{a} = [a^{L}, a^{M}, a^{U}]$ and $\hat{b} = [b^{L}, b^{M}, b^{U}] \in \Psi$, then we call

$$p_{\lambda}(\hat{a} \geqslant \hat{b}) = \lambda \frac{\min\{l_{a^-} + l_{b^-}, \max(a^{M} - b^{L}, 0)\}}{l_{a^-} + l_{b^-}} + (1-\lambda)\frac{\min\{l_{a^+} + l_{b^+}, \max(a^{U} - b^{M}, 0)\}}{l_{a^+} + l_{b^+}} \quad (1)$$

the possibility degree of $\hat{a} \geqslant \hat{b}$, where $\lambda \in [0,1]$, $l_{a^-} = a^{M} - a^{L}$, $l_{a^+} = a^{U} - a^{M}$, $l_{b^-} = b^{M} - b^{L}$ and $l_{b^+} = b^{U} - b^{M}$.

Note: λ varies according to the risk tolerance of the DM. The DM is risk-averse, if $\lambda > 0.5$; the DM presents a propensity for risk, if $\lambda < 0.5$; the DM is risk-neutral, if $\lambda = 0.5$. Especially, $p_{\lambda}(\hat{a} \geqslant \hat{b})$ is the pessimistic possibility degree of $\hat{a} \geqslant \hat{b}$, when $\lambda = 1$; $p_{\lambda}(\hat{a} \geqslant \hat{b})$ is the optimistic possibility degree of $\hat{a} \geqslant \hat{b}$, if $\lambda = 0$.

The following conclusions can be obtained easily.

Theorem 1 Let $\hat{a} = [a^{L}, a^{M}, a^{U}]$ and $\hat{b} = [b^{L}, b^{M}, b^{U}] \in \Psi$, then

1) $0 \leqslant p_{\lambda}(\hat{a} \geqslant \hat{b}) \leqslant 1$.

2) If $b^{U} \leqslant a^{L}$, then $p_{\lambda}(\hat{a} \geqslant \hat{b}) = 1$. Similarly, if $a^{U} \leqslant b^{L}$, then $p_{\lambda}(\hat{b} \geqslant \hat{a}) = 1$.

3) If $a^{U} \leqslant b^{L}$, then $p_{\lambda}(\hat{a} \geqslant \hat{b}) = 0$. Similarly, if $b^{U} \leqslant a^{L}$, then $p_{\lambda}(\hat{b} \geqslant \hat{a}) = 0$.

4) $p_{\lambda}(\hat{a} \geqslant \hat{b}) + p_{\lambda}(\hat{b} \geqslant \hat{a}) = 1$. Especially, $p_{\lambda}(\hat{a} \geqslant \hat{a}) = \frac{1}{2}$.

We let $N = \{1,2,\cdots,n\}$ and give the following operational laws related to triangular fuzzy numbers.

1) $\hat{a} \otimes \hat{b} = [a^{L}, a^{M}, a^{U}] \otimes [b^{L}, b^{M}, b^{U}] = [a^{L}b^{L}, a^{M}b^{M}, a^{U}b^{U}]$.

2) $(\hat{a})^{\mu} = [(a^{L})^{\mu}, (a^{M})^{\mu}, (a^{U})^{\mu}]$, where $\mu \geqslant 0$.

2 The Fuzzy Induced Ordered Weighted Geometric Averaging Operator

Yager[5] introduced the ordered weighted averaging (OWA) operators which provide a parameterized family of mean type aggregation operators. An important feature of these operators is the reordering step. During this step the arguments are ordered by their values. Yager[8] further introduced the IOWA operators in which the ordering of the arguments is induced by another variable called the order inducing variable. Specifically, they can aggregate decision information in environments which mix linguistic and numerical variables. Motivated by the idea, we introduce a fuzzy induced ordered weighted geometric operator.

Definition 2 g is called an FIOWGA operator, if

$$g(\langle u_1, \hat{a}_1\rangle, \cdots, \langle u_n, \hat{a}_n\rangle) = \hat{b}_1^{w_1} \otimes \hat{b}_2^{w_2} \otimes \cdots \otimes \hat{b}_{n}^{w_j} = \left[\prod_{j=1}^{n}(b_j^{L})^{w_j}, \prod_{j=1}^{n}(b_j^{M})^{w_j}, \prod_{j=1}^{n}(b_j^{U})^{w_j}\right]$$

where $\boldsymbol{w} = \{w_1, w_2, \cdots, w_n\}^{T}$ is the associated weighting vector of g, $w_j \in [0,1]$, $\sum_{j=1}^{n} w_j = 1$, $\langle u_i, a_i\rangle$ is called an FIOWGA pair, $\hat{b}_j$ is the $\hat{a}$ value of the FIOWGA pair having the j-th largest u value, and all of the $\hat{a}_i$ are triangular fuzzy numbers. u_i is the order inducing variable and $\hat{a}_i$ the argument variable.

Example Consider a collection of FIOWGA pairs $\langle u_i, \hat{a}_i\rangle$, where $\hat{a}_i$ is a triangular fuzzy number and u_i is a value drawn from space Θ. Let Θ be the space consisting of

Θ = {very small (VS), small (S), medium (M), big (B), very big (VB)}

on which the following ordering exists on the objects VB > B > M > S > VS.

Let $\langle$S, $\hat{a}_1\rangle$, $\langle$VB, $\hat{a}_2\rangle$, and $\langle$M, $\hat{a}_3\rangle$ be a collection of three FIOWGA pairs, where $\hat{a}_1 = [7, 7.5, 8]$, $\hat{a}_2 = [2, 2.5, 3]$ and $\hat{a}_3 = [9, 10, 11]$. We desire to aggregate using the weighting vector $\boldsymbol{w}$ where $w_1 = 0.3$, $w_2 = 0.5$, and $w_3 = 0.2$. Performing the ordering of the FIOWGA pairs with respect to the first component, we get the ordered FIOWGA pairs: $\langle$VB, $\hat{a}_2\rangle$, $\langle$M, $\hat{a}_3\rangle$, and $\langle$S, $\hat{a}_1\rangle$. From this we get an aggregated value

$$g(\langle u_1, \hat{a}_1\rangle, \langle u_2, \hat{a}_2\rangle, \langle u_3, \hat{a}_3\rangle) = [2, 2.5, 3]^{0.3} \otimes [9, 10, 11]^{0.5} \otimes [7, 7.5, 8]^{0.2} = [5.45, 6.23, 6.99]$$

If there is a tie between two FIOWGA pairs: $\langle u_i, \hat{a}_i\rangle$ and $\langle u_j, \hat{a}_j\rangle$ with respect to the order inducing variable, then the policy we shall follow is to replace the arguments of the tied FIOWGA pairs by the geometric average of the arguments of the tied pairs. Thus we replace $\langle u_i, \hat{a}_i\rangle$ and $\langle u_j, \hat{a}_j\rangle$ by $\langle u_i, (\hat{a}_i \otimes \hat{a}_j)^{1/2}\rangle$ and $\langle u_j, (\hat{a}_i \otimes \hat{a}_j)^{1/2}\rangle$, respectively. We note that if t items are tied, then we replace these by t replica's of their geometric average.

3 The Linguistic Scale

In a usual framework, there is a finite set of alternatives $X = \{x_1, x_2, \cdots, x_n\}$. As in the real world, when decision-making processes take place in an environment in which the information is not precisely known, the DM compares these alternatives with respect to a single criterion by the linguistic scale in the set L = {EG, VG, G, SG, F, SP, P, VP, EP},where EG = Extremely Good; VG = Very Good; G = Good; SG = Slightly Good; F = Fair; SP = Slightly Poor; P = Poor; VP = Very Poor; EP = Extremely Poor. The linguistic scale in L is used by the DM (s) to evaluate the performance of decisions versus qualitative objectives, and these are characterized by triangular fuzzy numbers on [1/9, 9] as follows:

$$\begin{aligned}&EG = [7,9,9],\ VG = [5,7,9],\ G = [3,5,7]\\&SG = [1,3,5],\ F = [1/3,1,3]\\&SP = [1/5,1/3,1]],\ P = [1/7,1/5,1/3]\\&VP = [1/9,1/7,1/5],\ EP = [1/9,1/9,1/7]\end{aligned}$$

where EG > VG > G > SG > F > SP > P > VP > EP.

4 An Approach to Ranking Alternatives

Based on the linguistic scale above and the FIOWGA operator, we present an approach to ranking alternatives in decision-making problems.

Step 1 For a decision-making problem, the DM compares all the alternatives with respect to a single criterion using the linguistic scale in set L, and constructs a fuzzy linguistic preference matrix $\hat{\boldsymbol{A}}$, where the diagonal elements in $\hat{\boldsymbol{A}}$ are expressed as "—", which means "undefined", and the other elements in $\hat{\boldsymbol{A}}$ are taken from set L.

Step 2 Utilize the FIOWGA operator to aggregate the preference information in the i-th line of the matrix $\hat{\boldsymbol{A}}$, and then get the preference degree $\hat{z}_j(j \in N)$ of the i-th alternative over all the other alternatives. The FIOWGA operator reflects the fuzzy majority calculating its weighting vector by means of a fuzzy linguistic quantifier[7]. In the case of a non-decreasing proportional quantifier Q, the weighting vector can be obtained by the following expression

$$w_k = Q(k/n) - Q((k-1)/n) \quad k \in N \quad (2)$$

where

$$Q(r) = \begin{cases} 0 & r < a \\ \dfrac{r-a}{b-a} & a \leqslant r \leqslant b \\ 1 & r > b \end{cases} \quad (3)$$

with $a, b, r \in [0,1]$. Some examples of proportional quantifiers are shown in Fig.1, where the parameters, (a, b) are (0.3, 0.8), (0, 0.5) and (0.5, 1), respectively (see Ref.[7]).

Step 3 As $\hat{z}_j(j \in N)$ are triangular fuzzy numbers, we compare each $\hat{z}_i$ with all the $\hat{z}_j(j \in N)$ by using formula (1), and let $p_{ij} = p_\lambda(\hat{z}_i \geqslant \hat{z}_j)$, then construct the complementary matrix $\boldsymbol{P} = (p_{ij})_{n \times n}$, where $p_{ij} \geqslant 0$, $p_{ij} + p_{ji} = 1$, $p_{ii} = \dfrac{1}{2}$, $i, j \in N$.

Step 4 Sum all the elements in each line of the matrix $\boldsymbol{P}$, i.e., let $p_i = \sum_{j=1}^{n} p_{ij}$, $i \in N$, then rank $\hat{z}_i$ $(i \in N)$ in descending order in accordance with the value of p_i. Thus, we get the ranking of $x_i(i \in N)$.

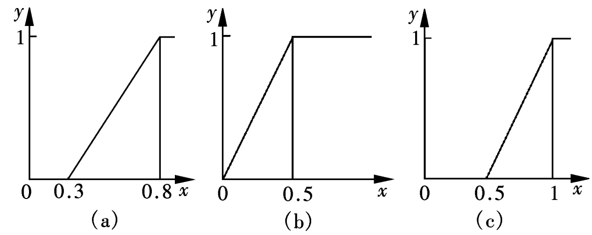

Fig.1 Proportional fuzzy quantifiers. (a) Most; (b) At least half; (c) As many as possible

5 Illustrative Example

In this section, a decision-making problem involves the evaluation of five schools $x_i(i = 1,\cdots,5)$ of a university. One main criterion used is research. The DM compares these five schools with respect to the criterion research by using the linguistic scale in set L, and constructs the fuzzy linguistic preference matrix $\hat{\boldsymbol{A}}$ as

$$\hat{\boldsymbol{A}} = \begin{bmatrix} — & G & SG & F & G \\ P & — & F & SP & VG \\ SP & F & — & P & EG \\ F & SG & G & — & VG \\ P & VP & EP & VP & — \end{bmatrix}$$

To rank these five schools $x_i(i = 1,\cdots,5)$, the following steps are involved.

Step 1 Use the fuzzy majority criterion with the fuzzy linguistic quantifier "at least half", with the pair $(a, b) = (0, 0.5)$, and by (2) and (3) we get the weighting vector $\boldsymbol{w} = \{0.4, 0.4, 0.2, 0\}^{\mathrm{T}}$.

Step 2 Utilize the FIOWGA operator to aggregate the preference information in the i-th line of the matrix $\hat{\boldsymbol{A}}$, and get the preference degree $\hat{z}_j(j = 1,\cdots,5)$ of the i-th alternative over all the other alternatives. To do so, we shall first get $\hat{z}_1$. Since $u_1 = G$, $\hat{a}_1 = [3, 5, 7]$; $u_2 = SG$, $\hat{a}_2 = [1, 3, 5]$; $u_3 = F$, $\hat{a}_3 = [1/3, 1, 3]$; $u_4 = G$, $\hat{a}_4 = [3, 5, 7]$, and $u_1 = u_4 > u_2 > u_3$, then $\hat{b}_1 = \hat{b}_2 = \hat{a}_1 = \hat{a}_4 = [3, 5, 7]$, $\hat{b}_3 = \hat{a}_2 = [1, 3, 5]$, $\hat{b}_4 = \hat{a}_3 = [1/3, 1, 3]$. Thus,

$$\begin{aligned}\hat{z}_1 &= g(\langle u_1, \hat{a}_1\rangle, \langle u_2, \hat{a}_2\rangle, \langle u_3, \hat{a}_3\rangle, \langle u_4, \hat{a}_4\rangle) = \\ &\hat{b}_1^{w_1} \otimes \hat{b}_2^{w_2} \otimes \hat{b}_3^{w_3} \otimes \hat{b}_4^{w_4} = [2.41, 4.51, 6.54]\end{aligned}$$

Similarly, we have

$$\hat{z}_2 = [0.89, 1.75, 3.74]$$
$$\hat{z}_3 = [1.02, 1.93, 3.74]$$
$$\hat{z}_4 = [2.95, 5.16, 7.24]$$
$$\hat{z}_5 = [0.12, 0.16, 0.25]$$

Step 3 Suppose that the DM is risk-neutral (λ

$= 0.5$), we utilize (1) to compare the $\hat{z}_j(j = 1,\cdots,5)$, and get the possibility degree matrix:

$$\boldsymbol{P} = \begin{bmatrix} 0.50 & 1 & 1 & 0.35 & 1 \\ 0 & 0.50 & 0.45 & 0 & 1 \\ 0 & 0.55 & 0.50 & 0 & 1 \\ 0.65 & 1 & 1 & 0.50 & 1 \\ 0 & 0 & 0 & 0 & 0.50 \end{bmatrix}$$

Step 4 Sum all the elements in each line of the matrix $\boldsymbol{P}$, it can be obtained that

$$p_1 = 3.85,\ p_2 = 1.95,\ p_3 = 2.05$$

$$p_4 = 4.15,\ p_5 = 0.50$$

then rank $\hat{z}_i(i = 1,\cdots,5)$ in descending order in accordance with the value of p_i, i.e., $\hat{z}_4 > \hat{z}_1 > \hat{z}_3 > \hat{z}_2 > \hat{z}_5$. Then, we get the ranking of $x_i(i = 1,\cdots,5)$ as follows: $x_4 \succ x_1 \succ x_3 \succ x_2 \succ x_5$.

Thus, the optimal school is x_4.

6 Conclusion

This paper has proposed an FIOWGA operator to aggregate linguistic preference information, and presented a method based on the linguistic scale and FIOWGA operator to rank alternatives in decision-making problems. The numerical result shows that the method is rational and effective.

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一种基于模糊语言标度和 FIOWGA 算子的决策方法

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摘　要　给出了一种用于方案比较的模糊语言标度，并用定义在区间[1/9，9]上的三角模糊数表示其数值含义. 利用该标度构造语言偏好矩阵，给出了一种集结语言偏好信息的模糊导出的有序加权几何平均(FIOWGA)算子，并提出了一种基于模糊语言标度和 FIOWGA 算子的决策方法. 最后，通过算例对方法的可行性和有效性进行了说明.

关键词　模糊语言标度，三角模糊数，FIOWGA 算子

中图分类号　C934