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On the growth of transcendental entire solutions of algebraic differential equations

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Abstract: In this paper, we investigate the growth of transcendental entire solutions of the following algebraic differential equation $a(z)f'^2 + (b_2(z)f^2 + b_1(z)f + b_0(z))f' = d_3(z)f^3 + d_2(z)f^2 + d_1(z)f + d_0(z)$, where $a(z)$, $b_i(z)$ $(0 \leqslant i \leqslant 2)$ and $d_j(z)$ $(0 \leqslant j \leqslant 3)$ are all polynomials, and this equation relates closely to the following well-known algebraic differential equation $C(z,w)w'^2 + B(z,w)w' + A(z,w) = 0$, where $C(z,w) \not\equiv 0$, $B(z,w)$ and $A(z,w)$ are three polynomials in z and w. We give relationships between the growth of entire solutions and the degrees of the above three polynomials in detail.

Key words: algebraic differential equation; degree; entire solutions

1 Introduction and Main Results

Firstly, we introduce some terminologies. Let f be a transcendental entire function and write

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

Then, use the following standard notations for maximum modulus $M(r,f)$, maximum term $\zeta(r,f)$, the central index $\nu(r,f)$, order $\rho(f)$ and lower order $\mu(f)$ of f, respectively.

$$M(r,f) = \max(|f(z)| : |z| = r)$$

$$\zeta(r,f) = \max_{|z|=r} |a_n z^n|$$

$$\nu(r,f) = \sup(n : |a_n| r^n = \zeta(r,f))$$

$$\rho = \rho(f) = \lim_{r\to\infty} \sup \frac{\log \log M(r,f)}{\log r}$$

$$\mu = \mu(f) = \lim_{r\to\infty} \inf \frac{\log \log M(r,f)}{\log r}$$

In this note, we investigate the growth of the entire solution of the following nonlinear algebraic differential equation:

$$a(z) f'^2 + (b_2(z)f^2 + b_1(z)f + b_0(z))f' = d_3(z)f^3 + d_2(z)f^2 + d_1(z)f + d_0(z) \tag{1}$$

where $a(z)$, $b_i(z)$ $(0 \leqslant i \leqslant 2)$ and $d_j(z)$ $(0 \leqslant j \leqslant 3)$ are all polynomials. Why do we only choose to consider this type of algebraic differential equations? The reason is Eq.(1) relates closely to the following algebraic differential equation:

$$C(z,w)\, w'^2 + B(z,w)w' + A(z,w) = 0 \tag{2}$$

where $C(z,w) \not\equiv 0$, $B(z,w)$ and $A(z,w)$ are three polynomials in z and w[1]. Ishizaki and Steinmetz[2,3] studied this type of algebraic differential equation in detail and got many interesting results. We refer readers to Refs.[4-6] for the introduction of complex differential equations theory.

In general, if we let $p(u_1, u_2, u_3)$ be a polynomial in all of its arguments u_1, u_2 and u_3, then the growth of the meromorphic solutions of the following algebraic differential equation are known:

$$p(z, f, f') = 0 \tag{3}$$

In the 1950's, Goldberg[7] proved the following theorem.

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Theorem 1 All the meromorphic solutions of Eq. (3) are of finite order of growth.

In 1983, Strelitz[8] got the following theorem.

Theorem 2 Every transcendental entire solution of a first-order algebraic differential equation with rational coefficients has an order no less than 1/2.

In 1996, Hayman[9] obtained the following result.

Theorem 3 Let $d = \max(\deg(a), \deg(b_0), \deg(b_1), \deg(b_2), \deg(d_0), \deg(d_1), \deg(d_2), \deg(d_3))$, and if $b_2(z) \not\equiv 0$, then any entire solution of (1) has finite order $\rho(f) \leqslant \max(2d, d+1)$.

Recently, Liao and Yang[1] showed the following thorem.

Theorem 4 If $\deg(d_2(z)) \neq \deg(a) - 1$ in (1) and $f(z)$ is a transcendental entire solution of Eq. (1), then $\rho(f) \geqslant 1$.

In this article, we shall show the following result.

Theorem 5 Let f be a transcendental entire solution of (1).

1) If $d_3(z) \not\equiv 0$ or $b_2(z) \not\equiv 0$, $\deg(d_3(z)) > \deg(b_2(z))$, then $\rho(f) \geqslant \deg(d_3(z)) - \deg(b_2(z)) + 1$;

2) If $d_3(z) \equiv 0$ and $b_2(z) \equiv 0$.

Subcase 1 If $a(z) \equiv 0$, $\deg(b_1(z)) < \deg(d_2(z))$, then $\rho(f) \geqslant \deg(d_2) - \deg(b_1) + 1$;

Subcase 2 If $d_2(z) \equiv 0$, $\deg(b_1(z)) > \deg(a(z))$, then $\rho(f) \geqslant \deg(b_1) - \deg(a) + 1$;

Subcase 3 If $b_1(z) \equiv 0$, $\deg(a(z)) < \deg(d_2(z))$, then $\rho(f) \geqslant (\deg(d_2) - \deg(a))/2 + 1$;

Subcase 4 If $a(z) \not\equiv 0$, $b_1(z) \not\equiv 0$ and $d_2(z) \not\equiv 0$, $\deg(b_1) = \max(\deg(a), \deg(b_1), \deg(d_2))$, then $\rho(f) \geqslant \deg(b_1) - \deg(a) + 1$;

Subcase 5 If $a(z) \not\equiv 0$, $b_1(z) \not\equiv 0$ and $d_2(z) \not\equiv 0$, $\deg(d_2) = \max(\deg(a), \deg(b_1), \deg(d_2))$, then $\rho(f) \geqslant \min((\deg(d_2) - \deg(a))/2 + 1, \deg(d_2) - \deg(b_1) + 1)$.

3) Otherwise $\mu(f) \leqslant 1$ and $\rho(f) \geqslant 1/2$.

2 Some Lemmas

Lemma 1 If f is an entire function of order ρ, then

$$\left.\begin{aligned} \rho &= \lim_{r\to\infty} \sup \frac{\log^+ \nu(r,f)}{\log r} = \lim_{r\to\infty} \sup \frac{\log^+ \log^+ \zeta(r,f)}{\log r} \\ \mu &= \lim_{r\to\infty} \inf \frac{\log^+ \nu(r,f)}{\log r} = \lim_{r\to\infty} \inf \frac{\log^+ \log^+ \zeta(r,f)}{\log r} \end{aligned}\right\} \tag{4}$$

Remark The proof of the first equality of the above lemma can be found in Ref. [4] and [5]. By using similar methods to those in Ref. [4], we can also prove the second one.

Lemma 2[5] Let f be a transcendental entire function, and let $0 < \delta < \frac{1}{4}$.

Suppose that at the point z with $|z| = r$, the inequality $|f(z)| > M(r,f)\nu(r,f)^{-(1/4)+\delta}$ holds. Then there exists a set $F \subset R^+$ of finite logarithmic measure, i.e., $\int dt/t < +\infty$, such that

$$f^{(m)}(z) = \left(\frac{\nu(r,f)}{z}\right)^m (1 + o(1))f(z) \tag{5}$$

holds for all $m \geqslant 0$ and all $r \notin F$.

3 Proof of Theorem 5

Using theorem 1 and lemma 1, we know that there exists a positive number $C > 1$ such that, for sufficiently large r,

$$\frac{\log^+ \nu(r,f)}{\log r} \leqslant C$$

so

$$\frac{\nu(r,f)}{r} \leqslant r^{C+1} \tag{6}$$

Now we choose $r_n \notin F$ and z_n such that

1) $r_n \to \infty$ as $n \to \infty$;

2) $|z_n| = r_n$, $|f(z_n)| = M(r_n, f)$;

3) $\lim\limits_{n\to\infty} \dfrac{f'(z_n)}{f(z_n)}$ exists.

From lemma 2, we have

$$\frac{f'(z_n)}{f(z_n)} = \frac{\nu(r_n, f)}{z_n}(1 + o(1)) \tag{7}$$

Now we rewrite (1) as

$$\left(d_3(z) - b_2(z)\frac{f'(z)}{f(z)}\right) = \left[a(z)\left(\frac{f'(z)}{f(z)}\right)^2 + b_1(z)\frac{f'(z)}{f(z)} + b_0(z)\frac{f'(z)}{(f(z))^2} - d_2(z) - \frac{d_1(z)}{f(z)} - \frac{d_0(z)}{(f(z))^2}\right]\frac{1}{f(z)} \tag{8}$$

From (6), (7), and (8), for any $n \geqslant 1$, we have

$$\left| a(z_n)\left(\frac{f'(z_n)}{f(z_n)}\right)^2 + b_1(z_n)\frac{f'(z_n)}{f(z_n)} + b_0(z_n)\frac{f'(z_n)}{(f(z_n))^2} - d_2(z_n) - \frac{d_1(z_n)}{f(z_n)} - \frac{d_0(z_n)}{(f(z_n))^2}\right| < Mr^{m'} \tag{9}$$

where M and m' are two positive constants.

Note the fact that $\lim\limits_{r\to\infty}\dfrac{\log M(r,f)}{r^t} = \infty$ for any natural number t, then from this, (8), and (9), we have

$$\lim_{n\to\infty}\left[d_3(z_n) - b_2(z_n)\frac{f'(z_n)}{f(z_n)}\right] = 0 \tag{10}$$

Two cases are to be considered.

Case 1 $d_3(z) \not\equiv 0$ or $b_2(z) \not\equiv 0$.

Subcase 1 $\deg(d_3(z)) < \deg(b_2(z))$.

In this subcase, without loss of generality, we can suppose that

$$d_3(z) = d_p z^p + d_{p-1}z^{p-1} + \cdots + d_1 z + d_0$$
$$b_2(z) = b_q z^q + b_{q-1}z^{q-1} + \cdots + b_1 z + b_0$$

Combining this with (10), we get

$$\left[d_p + d_{p-1}z_n^{-1} + \cdots + d_0 z_n^{-p} - (b_q z_n^{q-p} + \cdots + b_p + \cdots + b_1 z_n^{-p+1} + b_0 z_n^{-p})\frac{f'(z_n)}{f(z_n)}\right] \to 0$$

as $n \to \infty$, therefore

$$\lim_{n\to\infty}\frac{f'(z_n)}{f(z_n)} = 0$$

From this and (7), we have $\nu(r_n, f) \leqslant \varepsilon r_n$ for a sufficiently large n and for some positive constant $\varepsilon > 0$. So, by using lemma 1 and theorem 2, we get $\mu(f) \leqslant 1$ and $\rho(f) \geqslant 1/2$.

Subcase 2 $\deg(d_3(z)) = \deg(b_2(z))$.

One can see immediatedly that

$$\left[d_p + d_{p-1}z_n^{-1} + \cdots + d_0 z_n^{-p} - (b_p + b_{p-1}z_n^{-1} + \cdots + b_0 z^{-p})\frac{f'(z_n)}{f(z_n)}\right] \to 0$$

as $n \to \infty$, hence

$$\lim_{n\to\infty}\frac{f'(z_n)}{f(z_n)} = a \neq 0$$

for some constant a. It follows from this, and (7) that, for a sufficiently large n,

$$\nu(r_n, f) \leqslant (|a| + \varepsilon)r_n$$

here $\varepsilon(>0)$ is a constant, so, similarly we have $\mu(f) \leqslant 1$ and $\rho(f) \geqslant 1/2$.

Subcase 3 $\deg(d_3(z)) > \deg(b_2(z))$.

Then

$$\left[d_p + d_{p-1}z_n^{-1} + \cdots + d_0 z_n^{-p} - \left(b_q\frac{1}{z_n^{p-q}} + b_{q-1}\frac{1}{z_n^{p-q+1}} + \cdots + b_0\frac{1}{z_n^p}\right)\frac{f'(z_n)}{f(z_n)}\right] \to 0$$

as $n \to \infty$, and this implies that

$$\frac{f'(z_n)}{f(z_n)} \sim z_n^{p-q} \text{ as } n \to \infty$$

Therefore, from this, (7), and lemma 1, we have that

$$\rho(f) \geqslant p - q + 1$$

Case 2 $d_3(z) \equiv 0$ and $b_2(z) \equiv 0$ but $a(z) \not\equiv 0$ or $b_1(z) \not\equiv 0$ or $d_2(z) \not\equiv 0$.

Then (1) becomes

$$a(z)f'^2 + (b_1(z)f + b_0(z))f' = d_2(z)f^2 + d_1(z)f + d_0(z) \tag{11}$$

When substituting z with z_n in (11), we can rewrite (11) as

$$\left[a(z_n)\left(\frac{f'(z_n)}{f(z_n)}\right)^2 + b_1(z_n)\frac{f'(z_n)}{f(z_n)} - d_2(z_n)\right] = \left[d_1(z_n) + \frac{d_0(z_n)}{f(z_n)} - b_0(z_n)\frac{f'(z_n)}{f(z_n)}\right]\frac{1}{f(z_n)}$$

Similarly, we have

$$\lim_{n\to\infty}\left[a(z_n)\left(\frac{f'(z_n)}{f(z_n)}\right)^2 + b_1(z_n)\frac{f'(z_n)}{f(z_n)} - d_2(z_n)\right] = 0 \tag{12}$$

Subcase 1 $a(z) \equiv 0$.

In this subcase, when using the arguments as in case 1, we can get that

1) If $\deg(b_1(z)) = \deg(d_2(z))$, then $\mu(f) \leqslant 1$ and $\rho(f) \geqslant 1/2$;

2) If $\deg(b_1(z)) < \deg(d_2(z))$, then $\rho(f) \geqslant \deg(d_2) - \deg(b_1) + 1$;

3) If $\deg(b_1(z)) > \deg(d_2(z))$, then $\mu(f) \leqslant 1$ and $\rho(f) \geqslant 1/2$.

Subcase 2 $d_2(z) \equiv 0$.

In this subcase, similarly

1) If $\deg(b_1(z)) = \deg(a(z))$, then $\mu(f) \leqslant 1$ and $\rho(f) \geqslant 1/2$;

2) If $\deg(b_1(z)) > \deg(a(z))$, then $\rho(f) \geqslant \deg(b_1) - \deg(a) + 1$;

3) If $\deg(b_1(z)) < \deg(a(z))$, then $\mu(f) \leqslant 1$ and $\rho(f) \geqslant 1/2$.

Subcase 3 $b_1(z) \equiv 0$.

In this subcase, by using the same arguments as in case 1, we have that

1) If $\deg(a(z)) = \deg(d_2(z))$, then $\mu(f) \leqslant 1$ and $\rho(f) \geqslant 1/2$;

2) If $\deg(a(z)) < \deg(d_2(z))$, then $\rho(f) \geqslant (\deg(d_2) - \deg(a))/2 + 1$;

3) If $\deg(a(z)) > \deg(d_2(z))$, then $\mu(f) \leqslant 1$ and $\rho(f) \geqslant 1/2$.

Subcase 4 $a(z) \not\equiv 0$, $b_1(z) \not\equiv 0$ and $d_2(z) \not\equiv 0$.

1) If $\deg(a) = \max(\deg(a), \deg(b_1), \deg(d_2))$, then (12) yields $\lim_{n\to\infty}\frac{f'(z_n)}{f(z_n)} = 0$, so $\mu(f) \leqslant 1$ and $\rho(f) \geqslant 1/2$;

2) If $\deg(b_1) = \max(\deg(a), \deg(b_1), \deg(d_2))$, then (12) yields $\frac{f'(z_n)}{f(z_n)} \sim r_n^{\deg(b_1)-\deg(a)}$, so $\rho(f) \geqslant \deg(b_1) - \deg(a) + 1$;

3) If $\deg(d_2) = \max(\deg(a), \deg(b_1), \deg(d_2))$, then (12) yields

$$\frac{f'(z_n)}{f(z_n)} \sim r_n^{(\deg(d_2)-\deg(a))/2} \text{ or } \frac{f'(z_n)}{f(z_n)} \sim r_n^{\deg(d_2)-\deg(b_1)}$$

so

$$\rho(f) \geqslant (\deg(d_2) - \deg(a))/2 + 1 \text{ or } \rho(f) \geqslant \deg(d_2) - \deg(b_1) + 1$$

Now the proof of the theorem 5 is complete.

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关于代数微分方程的超越整解的增长性

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摘　要　研究了如下代数微分方程 $a(z)f'^2+(b_2(z)f^2+b_1(z)f+b_0(z))f'=d_3(z)f^3+d_2(z)f^2+d_1(z)f+d_0(z)$(这里 $a(z)$, $b_i(z)(0\leqslant i\leqslant 2)$和 $d_j(z)$ $(0\leqslant j\leqslant 3)$是多项式)超越整函数解的增长性,这类方程与有名的代数微分方程 $C(z,w)w'^2+B(z,w)w'+A(z,w)=0$($C(z,w)\not\equiv 0$, $B(z,w)$和 $A(z,w)$是 z 和 w 的 3 个多项式)有紧密的关系.详细地给出了第 1 个方程的整函数解的增长性与它的 3 个多项式的次数之间的关系.

关键词　代数微分方程; 次数; 整解

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