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Effect of phase noise in an OFDM/OQAM system

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Abstract: The performance of an OFDM/OQAM system under phase noise is analyzed. The analysis helps to direct the design of low cost tuners through specifying the required phase noise characteristics. Discrete time formulation of OFDM/OQAM is first derived with the square root raised cosine (SRRC) filter as the pulse-shaping filter. Then the effect of multiplicative phase noise is equivalently represented as additive white Gaussian noise (AWGN), the variance of which is given analytically. We can observe that the same result as OFDM/QAM system is derived. Lastly, all the analytical results are verified by the bit error rate (BER) degradation through Monte Carlo simulation.

Key words: OFDM/OQAM; OFDM/QAM; phase noise; BER performance

Orthogonal frequency division multiplexing (OFDM) is certainly, until now, the most important class of multicarrier modulation (MCM). The OFDM acronym often recovers two different types of modulation. In the first one, as proposed, for instance, in Ref.[1], each carrier is modulated using quadrature amplitude modulation (QAM). In this scheme, which is also called OFDM/QAM, QAM symbols are shaped with a rectangular window. In the second category of OFDM systems, which is also called orthogonally multiplexed QAM or OFDM with offset QAM (OFDM/OQAM), the modulation used for each subcarrier is a staggered offset QAM (OQAM). Both the OFDM/QAM and OFDM/OQAM modulation schemes theoretically guarantee orthogonality and a maximum and identical spectral efficiency. Furthermore, in practice, they can be implemented thanks to the DFT. An important difference comes from the fact that OFDM/OQAM allows the introduction of an efficient pulse shaping, which makes it less sensitive to frequency offset due to the transmission channel and to the receiver. Therefore, if OFDM/QAM constitutes the modulation kernel of the famous so-called coded OFDM (COFDM) system, OFDM/OQAM is now presented as a good candidate to get still higher bit rates over wireless channels[1].

Time-offset, frequency-offset, phase noise and PAPR are the open problems of OFDM systems. In Refs.[2,3], the effects of phase noise on the OFDM/QAM system are investigated. However, up to now, to our knowledge. No paper has been contributed to assess the impact of phase noise on the performance of OFDM/OQAM receiver. This paper presents our assessment and some conclusions.

This paper is organized as follows. In section 1, continuous-time formulation of OFDM/OQAM is reviewed and the discrete-time formulation is then derived. In section 2, a phase noise analysis in an OFDM/OQAM system is made. In section 3, the performance is evaluated. Simulation results are shown in section 4. The Conclusion is given in section 5.

Notations: $\mathrm{Re}\{c\}$ and $\mathrm{Im}\{c\}$ denote the real part and imaginary part of the complex-valued number c respectively. $L_2(\mathbf{R})$ and $l_2(\mathbf{Z})$ correspond to the space of squarable-summable continuous and discrete-time functions, respectively. $\| g \|$ is the L_2 norm of h. Superscript $*$ denotes complex conjugation.

1 Discrete-Time Formulation of OFDM/OQAM

In this section our goal is to obtain a discrete-time signal in $l_2(\mathbf{Z})$ with its classical real-valued product $\langle \cdot, \cdot \rangle_{l_2(\mathbf{Z}),\mathbf{R}}$, which is defined by

$$\forall (u,v) \in l_2(\mathbf{Z}) \times l_2(\mathbf{Z}), \langle v,u \rangle_{l_2(\mathbf{Z}),\mathbf{R}} = \mathrm{Re}\Big[\sum_{k=-\infty}^{+\infty} u^*(k) v(k)\Big] \tag{1}$$

Firstly, the continuous-time baseband transmitted signal can be written as[4]

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$$s(t) = \sum_{n=-\infty}^{+\infty}\sum_{m=0}^{M-1}[\mathrm{Re}(c_{2m,n})g(t-nT_0) + \mathrm{jIm}(c_{2m,n})g(t-nT_0-T_0/2)]\mathrm{e}^{\mathrm{j}2\pi(2m)F_0 t} + [\mathrm{jIm}(c_{2m+1,n})g(t-nT_0) + \mathrm{Re}(c_{2m+1,n})g(t-nT_0-T_0/2)]\mathrm{e}^{\mathrm{j}2\pi(2m)F_0 t} \quad (2)$$

where T_0 is the OFDM/OQAM symbol interval; $F_0 = 1/T_0$ is the spacing between two successive carriers; $c_{m,n}$ (zero mean and variance E_s) are the transmitted QAM symbols; $g(t)$ is the well known square root raised cosine (SRRC) pulse shaping filter with the rolloff factor equalling 0.22. The number of carriers, which is denoted by Q, is also assumed to be even $Q = 2M$. In order to get simplified and easier-to-manipulate expressions, the following notations can be introduced:

$$\left.\begin{array}{l} a_{2m,2n} = \mathrm{Re}(c_{2m,n}),\ a_{2m,2n+1} = \mathrm{Im}(c_{2m,n}),\ a_{2m+1,2n} = \mathrm{Im}(c_{2m,n}),\ a_{2m+1,2n+1} = \mathrm{Re}(c_{2m,n}) \\ \psi_{2m,2n} = 0,\ \psi_{2m,2n+1} = \pi/2,\quad \psi_{2m+1,2n} = \pi/2, \psi_{2m+1,2n+1} = 0 \\ \tau_0 = T_0/2 \\ \xi_{m,n}(t) = g(t-n\tau_0)\mathrm{e}^{\mathrm{j}2\pi m F_0 t}\mathrm{e}^{\mathrm{j}\psi_{m,n}} \end{array}\right\} \quad (3)$$

Then (1) is rewritten as

$$s(t) = \sum_{n=-\infty}^{+\infty}\sum_{m=0}^{2M-1} a_{m,n}\xi_{m,n}(t) \quad (4)$$

Thus the transmitted signal is like an expansion over some basis functions $\xi_{m,n}(t)$ with coefficients $a_{m,n}$.

Since the duration T_0 corresponds to the transmission of $2M$ complex-valued symbols, the critical sampling period is defined by

$$T_s = T_0/Q \quad (5)$$

This means that $s(t)$ is critically sampled and the spectral efficiency is still maximum. In order to get a causal discrete-time pulse shaping filter $g(k)$ with length equalling L, $g(t)$ is truncated to the interval $[-(L/2)T_s, (L/2)T_s]$ and is delayed by $[(L-1)/2]T_s$ time units. Moreover, in order to get a pulse shaping filter with a norm still approximately equalling 1, $g(k)$ has to be normalized with a multiplicative factor $\sqrt{T_s}$:

$$g(k) = \sqrt{T_s}\, g\left[\left(k - \frac{L-1}{2}\right)T_s\right] \quad (6)$$

Based on (3),(4) and the fact that $\tau_0 = MT_s$ and that $F_0\tau_0 = 1/2$, the baseband transmitted discrete-time OFDM/OQAM signal is such that

$$s(k) = \sum_{n=-\infty}^{+\infty}\sum_{m=0}^{2M-1} a_{m,n}\xi_{m,n}(k) \quad (7)$$

with

$$\xi_{m,n}(k) = g(k-nM)\mathrm{e}^{\mathrm{j}\frac{2\pi}{2M}m\left(k-\frac{L-1}{2}\right)}\mathrm{e}^{\mathrm{j}\psi_{m,n}} \quad (8)$$

At the demodulation stage, we get an estimate of the transmitted real-valued symbols using the real-valued inner product of $s(k)$ and $\xi_{m,n}(k)$:

$$\hat{a}_{m,n} = \langle s, \xi_{m,n}\rangle_{l_2(\mathbf{Z}),\mathbf{R}} = \mathrm{Re}\left[\sum_{k=-\infty}^{+\infty}\xi^*_{m,n}(k)s(k)\right] \quad (9)$$

Till now, functional block diagrams of a simplified OFDM/OQAM transmitter and receiver are shown in Fig.1, where the local oscillator (LO) is not perfect. Its output is usually degraded due to many factors, including short-term frequency drift that may in part be caused by temperature variations. The short-term frequency drift manifests itself as phase noise. It should be mentioned that the phase noise could be introduced by a combination of both Tx LO and Rx LO.

2 Phase Noise Analysis

Let $p(t)$ represent the continuous time phase process (real random process) of the local oscillator which will be assumed Gaussian with zero mean and power spectral density specified by some phase noise mask. As such, $p(t)$ can be written as

$$p(t) = h(t) \otimes n(t) \quad (10)$$

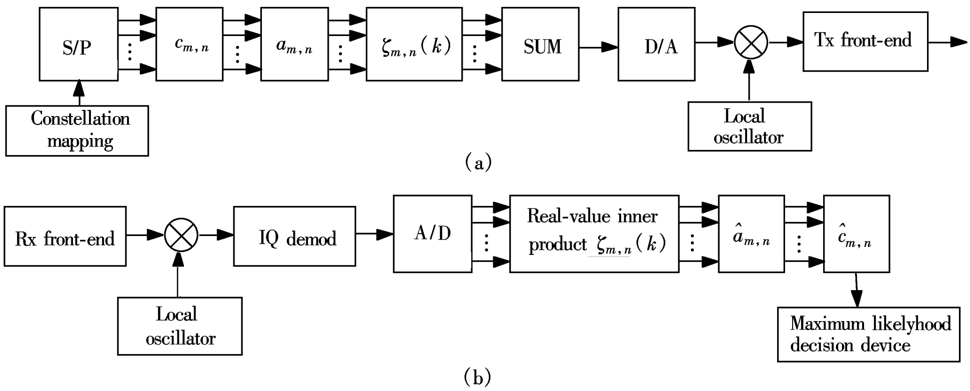

Fig.1 Block diagrams of an OFDM/OQAM transmitter and receiver. (a) OFDM/OQAM transmitter; (b) OFDM/OQAM receiver

where $h(t)$ is the impulse response of a low-pass linear filter whose frequency response is given by

$$N_0 \mid H(f) \mid^2 = P(f) \tag{11}$$

where $P(f)$ is the phase noise mask of the local oscillator; $n(t)$ is a white Gaussian noise process with power spectral density N_0. By definition, the autocorrelation function of the phase noise process is given by

$$R_p(\tau) = \int_{-\infty}^{\infty} P(f) \mathrm{e}^{\mathrm{j}2\pi f\tau} \mathrm{d}f \tag{12}$$

We are interested in the autocorrelation function of the sampled phase noise process. This is defined by

$$R_p(k, k') = E[p(kT_s) p^*(k'T_s)] \tag{13}$$

where

$$p(kT_s) = \int_{-\infty}^{\infty} h(\tau) n(kT_s - \tau) \mathrm{d}\tau \tag{14}$$

and therefore

$$\begin{aligned} R_p(k, k') &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau) h(\tau') E[n(kT_s - \tau) n(k'T_s - \tau')] \mathrm{d}\tau \mathrm{d}\tau' = \\ & N_0 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau) h(\tau') \delta(kT_s - k'T_s + \tau - \tau') \mathrm{d}\tau \mathrm{d}\tau' = \\ & N_0 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h[\tau - (k' - k) T_s] h(\tau) \mathrm{d}\tau \end{aligned} \tag{15}$$

From (17) and (14) the discrete autocorrelation function can be written in terms of the continuous phase noise power spectral density as

$$R_p(k, k') = \int_{-\infty}^{\infty} P(f) \mathrm{e}^{\mathrm{j}(kT_s - k'T_s)2\pi f} \mathrm{d}f = \int_{-\infty}^{\infty} P(f) \mathrm{e}^{\mathrm{j}(k-k')2\pi \frac{f}{f_s}} \mathrm{d}f = 2\int_{0}^{\infty} P(f) \cos 2\pi (k - k') \frac{f}{f_s} \mathrm{d}f \tag{16}$$

In order to simplify the analysis that follows, we make the assumptions: ① The communication channel is additive white Gaussian; ② The channel frequency response is flat; ③ The phase noise variance is very small compared to unity so that $\mathrm{e}^{\mathrm{j}p(t)} \approx 1 + \mathrm{j}p(t)$ can be invoked.

Subject to the above, in the receiving unit, the passband signal can be represented as $\mathrm{Re}\{s(t)\mathrm{e}^{\mathrm{j}\omega t + p(t)}\}$, where ω is the carrier frequency. After the In Phase and Quadrature Phase demodulation block, the received sampled baseband signal can be represented as

$$r(k) = s(k) + \mathrm{j}s(k)p(k) + n(k) \tag{17}$$

where $n(k)$ is complex additive white Gaussian noise with zero mean and variance N_0. Then using real inner product, we get an estimate of the transmitted real-symbol as

$$\begin{aligned} \hat{a}_{m,n} &= \langle r, \gamma_{m,n} \rangle_{l_2(\mathbf{Z}),\mathbf{R}} = \langle s, \gamma_{m,n} \rangle_{l_2(\mathbf{Z}),\mathbf{R}} + \langle \mathrm{j}sp, \gamma_{m,n} \rangle_{l_2(\mathbf{Z}),\mathbf{R}} + \langle n, \gamma_{m,n} \rangle_{l_2(\mathbf{Z}),\mathbf{R}} = \\ & a_{m,n} + \langle \mathrm{j}sp, \gamma_{m,n} \rangle_{l_2(\mathbf{Z}),\mathbf{R}} + n_{m,n} = a_{m,n} + e_{m,n} + n_{m,n} \end{aligned} \tag{18}$$

where $a_{m,n}$ is the desired signal, which is contaminated by real additive white Gaussian noise $n_{m,n}$ (with zero mean and power spectral density N_0) and error term $e_{m,n}$ induced by phase noise. In the following we will analyze more deeply the contribution of $e_{m,n}$. $e_{m,n}$ can be expanded as

$$e_{m,n} = \sum_{m'=0}^{2M-1} \sum_{n'=n-\frac{L-1}{M}}^{n+\frac{L-1}{M}} \langle \mathrm{j} a_{m',n'} p(k) \xi_{m',n'}(k), \xi_{m,n}(k) \rangle \tag{19}$$

Assuming that the $2M$ is relatively large and that $a_{m',n'}$ are statistically independent (zero mean and variance $0.5E_s$, i.e. $E(a_{m,n}) = 0$, and $E(a_{m,n}a_{m',n'}) = 0.5E_s\delta_{m,m'}\delta_{n,n'}$), it can be argued that the error term is a Gaussian process which is likely to affect the desired decision variables in the same way additive Gaussian does. The mean of $a_{m',n'}$ is obviously zero and the variance may be evaluated as

$$\sigma_e^2 = E(e_{m',n'}^2) = \sum_{m'=0}^{2M-1} \sum_{n'=n-\frac{L-1}{M}}^{n+\frac{L-1}{M}} E(a_{m',n'}^2) E[\langle \mathrm{j}p(k)\xi_{m',n'}(k), \xi_{m,n}(k)\rangle^2] =$$

$$\frac{1}{2}E_s \sum_{m'=0}^{2M-1} \sum_{n'=n-\frac{L-1}{M}}^{n+\frac{L-1}{M}} E[\langle \mathrm{j}p(k)\xi_{m',n'}(k), \xi_{m,n}(k)\rangle^2] = \frac{1}{2}E_s \sum_{m'=0}^{2M-1} \sum_{n'=n-\frac{L-1}{M}}^{n+\frac{L-1}{M}} \Phi_{m',n'}^2 \tag{20}$$

where

$$\begin{aligned}\Phi_{m',n'}^2 = &\sum_k \sum_{k'} E[p(k)p(k')] g(k-nM) g(k'-nM) g(k-n'M) g(k'-n'M) \cdot \\ &\sin\left[\frac{2\pi}{2M}(m'-m)\left(k-\frac{L-1}{2}\right) + \psi_{m',n'} - \psi_{m,n}\right] \cdot \\ &\sin\left[\frac{2\pi}{2M}(m'-m)\left(k'-\frac{L-1}{2}\right) + \psi_{m',n'} - \psi_{m,n}\right]\end{aligned} \tag{21}$$

and from (16) which can be rewritten as

$$\begin{aligned}\Phi_{m',n'}^2 = &2\int_0^{+\infty} P(f) \Big\{ \sum_k \sum_{k'} \cos\left[2\pi(k-k')\frac{f}{f_s}\right] \cdot g(k-nM) g(k'-nM) g(k-n'M) g(k'-n'M) \cdot \\ &\sin\left[\frac{2\pi}{2M}(m'-m)\left(k-\frac{L-1}{2}\right) + \psi_{m',n'} - \psi_{m,n}\right] \cdot \\ &\sin\left[\frac{2\pi}{2M}(m'-m)\left(k'-\frac{L-1}{2}\right) + \psi_{m',n'} - \psi_{m,n}\right]\Big\} \mathrm{d}f = 2\int_0^{+\infty} P(f) \Phi_{m',m,n',n}(f) \mathrm{d}f\end{aligned} \tag{22}$$

Therefore, we get

$$\sigma_e^2 = \frac{1}{2}E_s \sum_{m'=0}^{2M-1} \sum_{n'=n-\frac{L-1}{M}}^{n+\frac{L-1}{M}} \Phi_{m',n'}^2 = E_s \int_0^{+\infty} P(f) \sum_{m'} \sum_{n'} \Phi_{m',m,n',n}(f) \mathrm{d}f \tag{23}$$

where $P(f)$ is in reality a band-limited process with a bandwidth of less than half of the sampling rate. Therefore, the integration limits in (23) can be changed to the range $0 \cdots f_s/2$. With $f_d = f/f_s$, we rewrite (22) as

$$\sigma_e^2 = E_s \int_0^{1/2} P(f_d) \sum_{m'} \sum_{n'} \Phi_{m',m,n',n}(f_d) \mathrm{d}f_d \tag{24}$$

About (24) we have an assumption, which can be described as follows:

$\mathrm{SUM}_{\Phi(f_d,m,n)} = \sum_{m'} \sum_{n'} \Phi_{m',m,n',n}(f_d) \mathrm{d}f_d$ is constant during the range $0 \cdots f_s/2$ despite m and n.

It can be observed that if this assumption is true, the derivation of ${\sigma_e}^2$ will be greatly simplified. To prove this theoretically is very complex, however; fortunately, we can turn to the help of computer numerical computation to confirm this assumption. Without the loss of generality, only $n = 0$ or 1 is considered to verify the assumption.

Tab.1 shows the maximum and minimum values of $\mathrm{SUM}_{\Phi(f_d,m,n)}$ with respect to different m ($0 \leqslant m \leqslant 2M - 1$), n ($n = 0$ or 1), and different frequency points $f_d = \{\mathrm{I}/(2M), I = 0, 1 \cdots, M\}$. From Tab.1 we find the assumption is only closely related to the length of the pulse shaping filter. When L is sufficiently large, we can boldly assume the assumption is true. So, we can now summarize the important point in this section. The variance of the error term is

$$\sigma_e^2 = E_s \int_0^{\frac{1}{2}} P(f_d) \mathrm{d}f_d = \frac{1}{2}E_s\sigma_p^2 \tag{25}$$

where ${\sigma_p}^2$ is the variance of the phase noise.

Tab.1 The maximum and minimum value of $\mathrm{SUM}_{\Phi(f_d,m,n)}$

Q value	$L = 2Q$		$L = 4Q$		$L = 8Q$		$L = 16Q$	
	max	min	max	min	max	min	max	min
$Q = 32$	1.098 6	0.903 1	1.024 2	0.979 1	1.002 5	0.999 4	1.002 0	0.999 9
$Q = 128$	1.098 2	0.902 6	1.023 1	0.979 0	1.001 1	0.999 4	1.000 5	0.999 9
$Q = 512$	1.097 8	0.902 4	1.022 9	0.978 9	1.000 8	0.999 8	1.000 2	0.999 9
$Q = 1\,024$	1.097 8	0.902 2	1.022 9	0.978 8	1.000 4	0.999 8	1.000 1	0.999 9

3 Bit Error Rate (BER) Degradation

After recombination of $\hat{a}_{m,n}$ into QAM symbol $\hat{c}_{m,n}$, we get the estimated QAM symbol

$$\hat{c}_{m,n} = c_{m,n} + \tilde{e}_{m,n} + \tilde{n}_{m,n} \tag{26}$$

From (3) and (18), $\tilde{n}_{m,n}$ (zero mean and variance N_0) is a complex additive white Gaussian noise induced from the channel; $\tilde{e}_{m,n}$ (zero mean and variance $E_s\sigma_p^2$) is induced from phase noise and is to affect $c_{m,n}$ the same way as $\tilde{n}_{m,n}$ does.

The signal-to-noise-ratio (SNR) at the input of the decision device without phase noise is E_s/N_0. The SNR at the input device with phase noise is $E_s/(N_0 + E_s\sigma_p{}^2)$. The SNR loss in dB can be represented as

$$\mathrm{SNR}_{\mathrm{loss}} = 10\lg\left(\frac{E_s}{N_0}\right) - 10\lg\left(\frac{E_s}{N_0 + E_s\sigma_p{}^2}\right) = 10\lg\left(1 + \frac{E_s\sigma_p{}^2}{N_0}\right) \tag{27}$$

which is the same as OFDM/QAM system[2]. In fact, the BER performance under the Gaussian channel with different kinds of QAM modulations (such as QPSK, 16QAM, 64QAM) can be found in any digital communication textbook. Correspondingly, the theoretical BER degradation can be got easily by (27).

4 Simulation

The most common way of characterizing an oscillator's phase noise is with the single side band phase noise power density function, which represents the ratio (in dbc; "c" stands for carrier) between the single side band noise power in a 1 Hz bandwidth at a distance f from the carrier and the carrier power[5].

This characterization is normally performed by using a spectrum analyzer which provides the power spectral density of the equipment's phase noise N_{op} in relation to the carrier power C. From these measurements, given that the phase noise has a zero mean as we have assumed and it extends up to a frequency b (either because phase noise is band-limited or due to the presence of filtering in the receiver), its variance can be found as[4]

$$\sigma_p^2 = \int_0^b \frac{2N_{\mathrm{op}}}{C}\mathrm{d}f \tag{28}$$

Once its variance is found, (27) provides the SNR degradation introduced by phase noise.

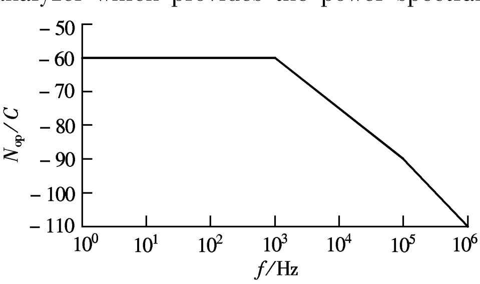

Fig.2 Phase noise spectrum mask ($s_p^2 = 0.015$ rad^2)

Fig.2 shows an example of a typical phase noise spectrum mask used in our simulation. The phase noise variance is $\sigma_p{}^2 = 0.015$ rad^2.

The sampling frequency defined in the simulation is $f_s = 3$ MHz, which means the subcarrier spacing $\Delta f = f_s/Q$ and the OFDM/OQAM symbol interval T_0 are

$$Q = 128,\ F_0 = 23.437\ \mathrm{kHz},\ T_0 = 42.67\ \mu\mathrm{s}$$

Modulated in QPSK, Fig.3 shows the result of BER performance under different situations. Being the truncation-induced nonorthogonal characteristic between $\xi_{m,n}(k)$, the BER performance under Gaussian noise is obtained through simulation. Then with (27) the theoretical BER with phase noise can be plotted. From Fig.3, we can observe the theoretical predicted result is very close to the simulated result, especially in the low SNR.

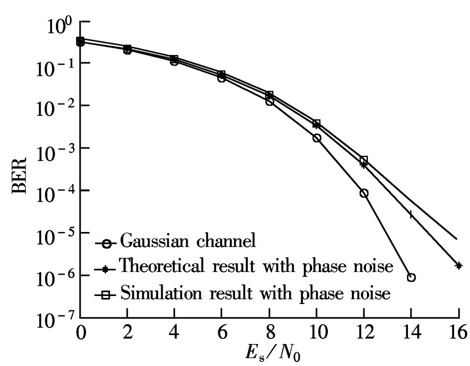

Fig.3 BER performance under different situations with $Q = 128$, $f_s = 3$ MHz

5 Conclusion

This paper presents an analytical procedure to quantify the impact of local oscillator phase noise on the performance of OFDM/OQAM systems over additive white Gaussian noise channels. With the SRRC filter as the pulse shaping filter, we can find that OFDM/OQAM and OFDM/QAM experienced the same performance loss induced from phase noise. It has been shown the simulation result is very consistent with the theoretical analysis.

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相位噪声对 OFDM/OQAM 系统的影响

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摘　要　分析了相位噪声对 OFDM/OQAM 系统性能的影响,分析的结果可用于指导设计低成本的调谐器.分析过程如下:首先导出了离散时间的采用 SRRC 成型滤波器的 OFDM/OQAM 系统公式;然后将乘性相位噪声等效表示为加性白高斯噪声,并给出了解析的方差表达式,该表达式与 OFDM/QAM 系统类似;最后利用 BER 的下降作为性能指标,采用蒙特卡罗仿真验证了我们的分析结果.

关键词　OFDM/OQAM; OFDM/QAM; 相位噪声; BER 性能

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