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Hybrid aggregation operator and its application to multiple attribute decision making problems

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Abstract: By combining the advantages of the additive weighted mean (AWM) operator and the ordered weighted averaging (OWA) operator, this paper first presents a hybrid operator for aggregating data information, and then proposes a hybrid aggregation (HA) operator-based method for multiple attribute decision making (MADM) problems. The theoretical analyses and the numerical results show that the HA operator generalizes both the AWM and OWA operators, and reflects the importance of both the given argument and the ordered position of the argument. Thus, the HA operator can reflect better real situations in practical applications. Finally, an illustrative example is given.

Key words: multiple attribute decision making; aggregation; operator

With the development of human society, how to aggregate and process the given data effectively is a very important issue in many fields, such as economics, management, the military, etc. People have put a premium on research on the aggregation operators[1-3]. The additive weighted mean (AWM) operator[1] is a classical aggregation operator, which is used to combine arguments (data) according to a set of weights. The fundamental aspect of the AWM operator is to weight the argument directly. The ordered weighted averaging (OWA) operator was introduced by Yager[3] to provide for aggregation lying between the max and min operators. Its fundamental aspect is the re-ordering step in which the arguments are rearranged in descending order. It is noteworthy that the OWA operator weights the ordered position of the argument instead of weighting the argument itself. Recently, the OWA operators have been investigated in many documents[3-10], and used in an astonishingly wide range of applications including decision-making, neural networks, database systems, fuzzy logic controllers, expert systems, market research, mathematical programming, lossless image compression[4], etc. In this paper, by combining the advantages of the AWM operator and the OWA operator, we present a hybrid aggregation (HA) operator, and then propose an HA operator-based method for MADM problems.

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1 AWM Operator and OWA Operator

For simplicity, we let $M = \{1,2,\cdots,m\}$ and $N = \{1,2,\cdots,n\}$.

Definition 1[1] Let AWM: $\mathbf{R}^n \rightarrow \mathbf{R}$, if $\mathrm{AWM}_{\boldsymbol{\omega}}(a_1, a_2, \cdots, a_n) = \sum_{j=1}^{n} \omega_j a_j$, where $\boldsymbol{\omega} = \{\omega_1, \omega_2, \cdots, \omega_n\}^{\mathrm{T}}$ is the weighting vector of a_i, with $\omega_j \in [0,1]$ and $\sum_{j=1}^{n} \omega_j = 1$, then the function AWM is called the AWM operator of dimension n.

The AWM operator has the following properties[1,5]:

1) Monotonicity Let $(a_1, a_2, \cdots, a_n)$ and $(\hat{a}_1, \hat{a}_2, \cdots, \hat{a}_n)$ be two collections of arguments, if $a_i \leqslant \hat{a}_i$, for any i, then

$$\mathrm{AWM}_{\boldsymbol{\omega}}(a_1, a_2, \cdots, a_n) \leqslant \mathrm{AWM}_{\boldsymbol{\omega}}(\hat{a}_1, \hat{a}_2, \cdots, \hat{a}_n)$$

2) Idempotency Let $(a_1, a_2, \cdots, a_n)$ be a collection of arguments, if $a_i = a$, for any i, then

$$\mathrm{AWM}_{\boldsymbol{\omega}}(a_1, \cdots, a_n) = a$$

3) Bounded The AWM operator lies between the max and min operators:

$$\min_i(a_i) \leqslant \mathrm{AWM}_{\boldsymbol{\omega}}(a_1, a_2, \cdots, a_n) \leqslant \max_i(a_i)$$

4) If $\boldsymbol{\omega} = \left\{\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right\}^{\mathrm{T}}$, the AWM operator is reduced to the arithmetic average operator:

$$\mathrm{AWM}_{\boldsymbol{\omega}}(a_1, a_2, \cdots, a_n) = \frac{1}{n}\sum_{i=1}^{n} a_i$$

Definition 2[3] Let OWA: $\mathbf{R}^n \rightarrow \mathbf{R}$, if $\mathrm{OWA}_{\boldsymbol{w}}(a_1, a_2, \cdots, a_n) = \sum_{j=1}^{n} w_j b_j$, where $\boldsymbol{w} = \{w_1, w_2, \cdots, w_n\}^{\mathrm{T}}$ is the associated weighting vector, with

$w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, and b_j is the j-th largest element in the set $\{a_1, a_2, \cdots, a_n\}$, then the function OWA is called the ordered OWA operator of dimension n.

The fundamental aspect of the OWA operator is the re-ordering step. In particular, an argument a_i is not associated with a particular weight w_i, but rather a weight w_i is associated with a particular ordered position i of the arguments (therefore, the weighting vector $\boldsymbol{w}$ is also called the position vector).

The OWA operator has the following properties[3]:

1) Commutativity Let $(a_1, a_2, \cdots, a_n)$ be a collection of arguments, and $(a_1', a_2', \cdots, a_n')$ be any permutation of $(a_1, a_2, \cdots, a_n)$, then

$$\mathrm{OWA}_{\boldsymbol{w}}(a_1, a_2, \cdots, a_n) = \mathrm{OWA}_{\boldsymbol{w}}(a_1', a_2', \cdots, a_n')$$

2) Idempotency Let $(a_1, a_2, \cdots, a_n)$ be a collection of arguments, if $a_i = a$, for any i, then

$$\mathrm{OWA}_{\boldsymbol{w}}(a_1, a_2, \cdots, a_n) = a$$

3) Monotonicity Let $(a_1, a_2, \cdots, a_n)$ and $(\hat{a}_1, \hat{a}_2, \cdots, \hat{a}_n)$ be two collections of arguments, if $a_i \leqslant \hat{a}_i$, for any i, then

$$\mathrm{OWA}_{\boldsymbol{w}}(a_1, a_2, \cdots, a_n) \leqslant \mathrm{OWA}_{\boldsymbol{w}}(\hat{a}_1, \hat{a}_2, \cdots, \hat{a}_n)$$

4) Bounded The OWA operator lies between the max and min operators:

$$\min_i(a_i) \leqslant \mathrm{OWA}_{\boldsymbol{w}}(a_1, a_2, \cdots, a_n) \leqslant \max_i(a_i)$$

5) If $\boldsymbol{w} = \left\{\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right\}^{\mathrm{T}}$, the OWA operator is reduced to the arithmetic average operator:

$$\mathrm{OWA}_{\boldsymbol{w}}(a_1, a_2, \cdots, a_n) = \frac{1}{n}\sum_{i=1}^{n} a_i$$

6) If $\boldsymbol{w} = \{1, 0, \cdots, 0\}^{\mathrm{T}}$, the OWA operator is reduced to the max operator:

$$\mathrm{OWA}_{\boldsymbol{w}}(a_1, a_2, \cdots, a_n) = \max_i(a_i)$$

7) If $\boldsymbol{w} = \{0, 0, \cdots, 1\}^{\mathrm{T}}$, the OWA operator is reduced to the min operator:

$$\mathrm{OWA}_{\boldsymbol{w}}(a_1, a_2, \cdots, a_n) = \min_i(a_i)$$

2 HA Operator

Definition 3 Let HA: $\mathbf{R}^n \to \mathbf{R}$, if $\mathrm{HA}_{\boldsymbol{\omega}, \boldsymbol{w}}(a_1, a_2, \cdots, a_n) = \sum_{j=1}^{n} w_j b_j$, where $\boldsymbol{w} = \{w_1, w_2, \cdots, w_n\}^{\mathrm{T}}$ is the associated weighting vector, with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, where b_j is the j-th largest of the weighted arguments $n\omega_i a_i (i \in N)$, $\boldsymbol{\omega} = \{\omega_1, \omega_2, \cdots, \omega_n\}^{\mathrm{T}}$ is the weighting vector of the $a_i (i \in N)$, with $\omega_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, and n is the balancing coefficient, then the function HA is called the HA operator of dimension n.

Theorem 1 The AWM operator is a special case of the HA operator.

Proof Let $\boldsymbol{w} = \{1/n, 1/n, \cdots, 1/n\}^{\mathrm{T}}$, then

$$\mathrm{HA}_{\boldsymbol{\omega}, \boldsymbol{w}}(a_1, a_2, \cdots, a_n) = \sum_{j=1}^{n} w_j b_j = \sum_{j=1}^{n} \frac{1}{n} b_j = \sum_{i=1}^{n} \omega_i a_i = \mathrm{AWM}_{\boldsymbol{\omega}}(a_1, a_2, \cdots, a_n)$$

Theorem 2 The OWA operator is a special case of the HA operator.

Proof Let $\boldsymbol{\omega} = \left\{\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right\}^{\mathrm{T}}$, then $n\omega_i a_i = a_i$, $i \in N$, therefore,

$$\mathrm{HA}_{\boldsymbol{\omega}, \boldsymbol{w}}(a_1, a_2, \cdots, a_n) = \mathrm{OWA}_{\boldsymbol{w}}(a_1, a_2, \cdots, a_n)$$

which completes the proof of theorem 2.

From theorems 1 and 2, we know that the HA operator generalizes both the AWM and OWA operators, and reflects the importance of both the given argument and the ordered position of the argument. Thus, the HA operator can reflect better real situations in practical applications.

3 An HA Operator-Based Method in MADM

For an MADM problem, let $X = \{x_1, x_2, \cdots, x_m\}$ be a discrete set of alternatives, $U = \{u_1, u_2, \cdots, u_n\}$ be a set of attributes, and $\boldsymbol{\omega} = \{\omega_1, \omega_2, \cdots, \omega_n\}^{\mathrm{T}}$ be the weight vector of the attributes, where $\omega_j \geqslant 0$, $\forall j$ and $\sum_{j=1}^{n} w_j = 1$. Let $\boldsymbol{A} = (a_{ij})_{m \times n}$ be the decision matrix, where a_{ij} is a numerical attribute value for alternative x_i with respect to attribute u_j.

In general, there are benefit attributes, fixation attributes, interval attributes, deviation attributes, deviated interval attributes, and cost attributes in MADM problems, and the “dimensions” of different attributes may be different. In order to measure all attributes in dimensionless units, we need to normalize the attribute values. Suppose that each attribute value a_{ij} in matrix $\boldsymbol{A} = (a_{ij})_{m \times n}$ is normalized into a corresponding element in matrix $\boldsymbol{R} = (r_{ij})_{m \times n}$.

In the following, we give an HA operator-based method for MADM problems. The procedure is described as follows.

Step 1 Utilize the 0.1 – 0.9 complementary scale[11] to compare each of the two attributes, and construct the fuzzy complementary matrix $\boldsymbol{P} = (p_{ij})_{n \times n}$, where $p_{ij} \geqslant 0$, $\forall i$, $p_{ij} + p_{ji} = 1$, and $p_{ii} = 0.5$, and then utilize the simple priority formula of the fuzzy complementary matrix (proposed in Ref.[12]):

$$\omega_i = \frac{1}{n(n-1)}\left(\sum_{j=1}^{n} p_{ij} + \frac{n}{2} - 1\right) \qquad i \in N \quad (1)$$

to get the attribute weight vector $\boldsymbol{\omega} = \{\omega_1, \omega_2, \cdots, \omega_n\}^{\mathrm{T}}$.

Step 2 Utilize the decision information given in matrix $\boldsymbol{R}$ and the HA operator:

$$z_i = \mathrm{HA}_{\boldsymbol{\omega},\boldsymbol{w}}(r_{i1}, r_{i2}\cdots, r_{in}) = \sum_{j=1}^{n} w_j b_{ij} \quad i \in M \quad (2)$$

to derive the overall value z_i of the alternative x_i ($i \in M$), where b_{ij} is the j-th largest of the weighted arguments $n\omega_l a_{il}$ ($l \in N$) corresponding to the alternative x_i, $\boldsymbol{\omega} = \{\omega_1, \omega_2, \cdots, \omega_n\}^{\mathrm{T}}$ is the attribute weight vector of the a_i ($i \in N$), with $\omega_j \in [0,1]$ and $\sum_{j=1}^{n} \omega_j = 1$, and n is the balancing coefficient. In the case of a non-decreasing proportional quantifier Q, the weighting vector $\boldsymbol{\omega} = \{w_1, w_2, \cdots, w_n\}^{\mathrm{T}}$ associated with the HA operator can be obtained by using the following expression[3, 13]:

$$w_j = Q(j/n) - Q[(j-1)/n] \qquad j \in N \quad (3)$$

where

$$Q(r) = \begin{cases} 0 & r < a \\ (r-a)/(b-a) & a \leqslant r \leqslant b \\ 1 & r > b \end{cases} \quad (4)$$

with $a, b, r \in [0,1]$. Some examples of proportional quantifiers are shown in Fig.1, where the parameters (a, b) are $(0.3, 0.8)$, $(0, 0.5)$, and $(0.5, 1)$, respectively.

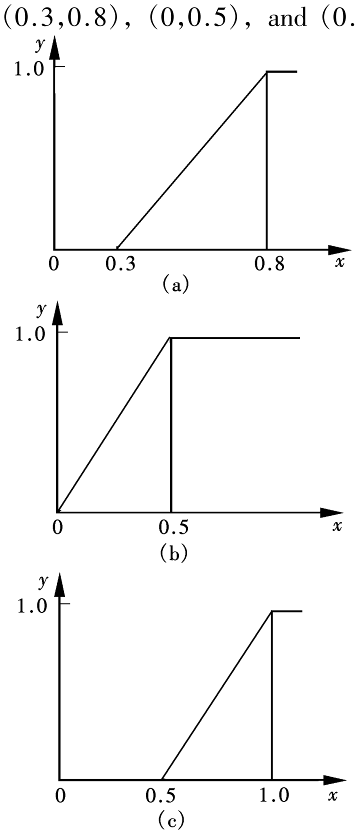

Fig.1 Proportional fuzzy quantifiers. (a) Most; (b) At least; (c) As many as possible

Step 3 Utilize z_i ($i \in M$) to rank the alternatives and then to select the best one(s).

Step 4 End.

4 Illustrative Example

In this section, an MADM problem of selecting a robot[14] is used to illustrate the proposed approach.

A robot user intends to select a robot and there are four alternatives for him/her to choose. When making a decision, the attributes considered include: ① p_1: cost (\$10 000); ② p_2: velocity (m/s); ③ p_3: repeatability (mm); ④ p_4: load capacity (kg). Among four attributes, p_2 and p_4 are of benefit type, p_1 and p_3 are of cost type. The decision information about robots is presented in Tab.1.

Tab.1 The decision information about robots

Alternative	Attribute			
	u_1	u_2	u_3	u_4
x_1	3.0	1.0	1.0	70
x_2	2.5	0.8	0.8	50
x_3	1.8	0.5	2.0	110
x_4	2.2	0.7	1.2	90

Step 1 According to Tab.1, the decision matrix $\boldsymbol{A}$ for the MADM problem is

$$\boldsymbol{A} = \begin{bmatrix} 3.0 & 1.0 & 1.0 & 70 \\ 2.5 & 0.8 & 0.8 & 50 \\ 1.8 & 0.5 & 2.0 & 110 \\ 2.2 & 0.7 & 1.2 & 90 \end{bmatrix}$$

which can be normalized into matrix $\boldsymbol{R}$ by using the formulae:

$$r_{ij} = \frac{a_{ij}}{\max_i a_{ij}} \qquad i = 1,2,3,4;\ j = 2,4$$

$$r_{ij} = \frac{\min_i a_{ij}}{a_{ij}} \qquad i = 1,2,3,4;\ j = 1,3$$

where

$$\boldsymbol{R} = \begin{bmatrix} 0.6 & 1 & 0.8 & 0.636 \\ 0.72 & 0.8 & 1 & 0.455 \\ 1 & 0.5 & 0.4 & 1 \\ 0.818 & 0.7 & 0.667 & 0.818 \end{bmatrix}$$

Step 2 Utilize the 0.1-0.9 complementary scale to compare each of the two attributes, and construct the fuzzy complementary matrix

$$\boldsymbol{P} = \begin{bmatrix} 0.5 & 0.3 & 0.4 & 0.2 \\ 0.7 & 0.5 & 0.6 & 0.4 \\ 0.6 & 0.4 & 0.5 & 0.4 \\ 0.8 & 0.6 & 0.6 & 0.5 \end{bmatrix}$$

By using Eq.(1), we can get the attribute weight vector $\boldsymbol{\omega} = \{0.200, 0.267, 0.242, 0.292\}^{\mathrm{T}}$.

Step 3 Use the fuzzy linguistic quantifier "most", with the pair (0, 0.5), and by (3) and (4), we get the weighting vector (associated with the HA operator) $\boldsymbol{w} = \{0.4, 0.4, 0.2, 0\}^{\mathrm{T}}$.

Step 4 Utilize the decision information given in matrix $\boldsymbol{R}$ and the HA operator

$$z_i = \mathrm{HA}_{\boldsymbol{\omega},\boldsymbol{w}}(r_{i1}, r_{i2}, r_{i3}, r_{i4}) = \sum_{j=1}^{4} w_j b_{ij} \qquad i = 1,2,3,4$$

to get the overall value z_i of the alternative x_i:

$$z_1 = 0.8855,\ z_2 = 0.8442,\ z_3 = 0.8940,\ z_4 = 0.8121$$

Step 5 Utilize $z_i(i = 1,2,3,4)$ to rank the alternatives as follows:

$$z_3 > z_1 > z_2 > z_4$$

and thus the best alternative is x_3.

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一种混合集结算子及其在多属性决策中的应用

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摘　要　结合加性加权平均(AWM)算子和有序加权平均(OWA)算子的特点，提出了一种集结决策信息的混合集结(HA)算子，并提出了一种基于混合集结(HA)算子的多属性决策方法．理论分析和数值结果表明：混合集结(HA)算子同时推广了加性加权平均(AWM)算子和有序加权平均(OWA)算子，它不仅能反映所给数据自身的重要性程度，而且还体现了数据所在位置的重要性程度．因此，混合集结(HA) 算子在实际应用中能更好地反映现实情况．最后进行了实例分析．

关键词　多属性决策；集结；算子

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