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Simultaneous diagonalization of two quaternion matrices

Zhou Jianhua

(Department of Mathematics, Southeast University, Nanjing 210096, China)

Abstract: The simultaneous diagonalization by congruence of pairs of Hermitian quaternion matrices is discussed. The problem is reduced to a parallel one on complex matrices by using the complex adjoint matrix related to each quaternion matrix. It is proved that any two semi-positive definite Hermitian quaternion matrices can be simultaneously diagonalized by congruence.

Key words: semi-positive definite matrix; quaternion matrix; adjoint matrix; congruence

The system of quaternions was introduced by Hamilton in 1843 as an example of non-commutative division algebras. It plays an important role in the classification of finite dimensional associative algebras. The study on the quaternion matrices can date back to the work of L. A. Wolf in 1936[1]. For its background in the study of quantum physics[2,3], in recent years, much work on this topic has been done. A theory has been well developed similar to the case of complex matrices. In spite of the difficulties caused by the non-commutativity of the multiplication of the quaternions, one has obtained a lot of parallel conclusions for the quaternion case (on the determinant, rank, eigenvalues, relations of similarity and congruence, etc.). One can find a systematical discussion in Ref. [4]. In this short paper, we will discuss the problem of simultaneous diagonalization of pairs of Hermitian quaternion matrices. We will prove the following main theorems:

Theorem Let $\boldsymbol{A}, \boldsymbol{B}$ be two $n \times n$ Hermitian quaternion matrices. If $\boldsymbol{A}, \boldsymbol{B}$ are all semi-positive definite, then there exists an $n \times n$ invertible matrix $\boldsymbol{C}$ such that $\boldsymbol{C}^* \boldsymbol{A}\boldsymbol{C}, \boldsymbol{C}^* \boldsymbol{B}\boldsymbol{C}$ are all diagonal (where $\boldsymbol{C}^*$ denotes the conjugate transpose of $\boldsymbol{C}$ defined in the below).

One can prove that any two semi-positive definite complex matrices can be simultaneously diagonalized by congruence. So the theorem can be considered as a generalization of this result to the case of quaternion matrices.

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Biography: Zhou Jianhua (1962—), male, associate professor, jhzhou@seu.edu.cn.

1 Preliminaries

For convenience, we recall some notions and results on the quaternions and the quaternion matrices.

Let Q be the algebra of real quaternion and e, i, j, k be the canonical basis of Q. Then, a quaternion $x = x_0 e + x_1 i + x_2 j + x_3 k$ with real numbers x_0, x_1, x_2, x_3 is often simply written as $x = x_0 + x_1 i + x_2 j + x_3 k$. For any $x = x_0 + x_1 i + x_2 j + x_3 k \in Q$, we call x_0, $x_0 + x_1 i$ and $x_1 i + x_2 j + x_3 k$ the real part, the complex part and the imaginary part of x, respectively. Let $\bar{x} = x^* = x_0 - x_1 i - x_2 j - x_3 k$ be the conjugate of x, then xx^* is always a real number for all $x \in Q$.

Let $\boldsymbol{A} = (a_{i,j})_{n\times n}$ be an $n \times n$ quaternion matrix. Denote by $A^* = (a^*_{j,i})_{n\times n}$ the conjugate transpose of $\boldsymbol{A}$. If there exists an $n \times n$ quaternion matrix $\boldsymbol{B}$ such that $\boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{I}$, the identity matrix, then $\boldsymbol{A}$ is called invertible, and $\boldsymbol{B}$ is called the inverse of $\boldsymbol{A}$. If $\boldsymbol{A}^*$ is the inverse of $\boldsymbol{A}$, then $\boldsymbol{A}$ is called unitary. If $\boldsymbol{A} = \boldsymbol{A}^*$, then $\boldsymbol{A}$ is called Hermitian. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are both Hermitian, then we say that $\boldsymbol{A}$ is congruent to $\boldsymbol{B}$ if there exists an invertible matrix $\boldsymbol{C}$ such that $\boldsymbol{B} = \boldsymbol{C}^* \boldsymbol{A}\boldsymbol{C}$.

It has been shown that Schur's lemma holds also for the quaternion matrices[4,5]: for any $n \times n$ quaternion matrix $\boldsymbol{A}$, there exists a unitary matrix $\boldsymbol{U}$, such that $\boldsymbol{U}^* \boldsymbol{A}\boldsymbol{U}$ is an upper triangular matrix. So, if $\boldsymbol{A}$ is Hermitian, then the upper triangular matrix $\boldsymbol{U}^* \boldsymbol{A}\boldsymbol{U}$ in Schur's lemma is in fact a real diagonal one. Therefore, any Hermitian quaternion matrix is congruent to a real diagonal matrix.

If $\boldsymbol{A}$ is an $n \times n$ Hermitian quaternion matrix, then $\boldsymbol{\eta}^* \boldsymbol{A}\boldsymbol{\eta}$ is a real number for any n dimensional column vector $\boldsymbol{\eta}$ with quaternion components. For a

Hermitian quaternion matrix $\boldsymbol{A}$, $\boldsymbol{A}$ is called positive definite (semi-positive definite, respectively) if $\boldsymbol{\eta}^* \boldsymbol{A}\boldsymbol{\eta}$ is always positive (nonnegative) for any nonzero $\boldsymbol{\eta}$. Similar to the case of complex matrices, one can easily prove the following results:

1) If $\boldsymbol{A}$ is positive definite (semi-positive definite), then the elements on the diagonal line of $\boldsymbol{A}$ are all positive (non-negative);

2) If $\boldsymbol{A}$ is diagonal, then $\boldsymbol{A}$ is positive definite (semi-positive definite) if and only if the elements on the diagonal line of $\boldsymbol{A}$ are all positive (non-negative);

3) If Hermitian matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are congruent, then $\boldsymbol{A}$ is positive definite (semi-positive definite) if and only if $\boldsymbol{B}$ is so;

4) Hermitian quaternion matrix $\boldsymbol{A}$ is semi-positive definite if and only if $\boldsymbol{A}$ is congruent to a matrix of the form $\begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{bmatrix}$.

There is a theory on quaternion matrices similar to the case of complex matrices. The main difficulties in the study of the quaternion matrices are caused by the non-commutativity of the multiplication of the quaternions. So the keys are often to find ways of passing the problems on quaternion matrices to ones on complex matrices.

The rank of a quaternion matrix $\boldsymbol{A}$ is defined as the maximum number of columns of $\boldsymbol{A}$ which are right linearly independent, and is denoted by $r(\boldsymbol{A})$. It can be shown that the product of $\boldsymbol{A}$ with any invertible matrices have the same rank as $\boldsymbol{A}$, and a square quaternion matrix $\boldsymbol{A}$ is invertible if and only if $r(\boldsymbol{A})$ is equal to its order[4].

A quaternion λ is called an eigenvalue (more exactly, a right eigenvalue) of matrix $\boldsymbol{A}$ provided that there is a nonzero column vector $\boldsymbol{\eta}$ such that $\boldsymbol{A}\boldsymbol{\eta} = \boldsymbol{\eta}\lambda$. A complex eigenvalue of $\boldsymbol{A}$ is said to be standard if it has a nonnegative imaginary part. One knows that any $n \times n$ quaternion matrix has exactly n standard eigenvalues (see theorem 5.4 of Ref.[4]).

Clearly, for every $x \in Q$, there exist complex numbers x_1, x_2, such that $x = x_1 + x_2 j$. So, for any quaternion matrix $\boldsymbol{A}$, there exist complex matrices $\boldsymbol{A}_1$, $\boldsymbol{A}_2$, such that $\boldsymbol{A} = \boldsymbol{A}_1 + \boldsymbol{A}_2 j$. Now, suppose that $\boldsymbol{A}$ is an $n \times n$ quaternion matrix and $\boldsymbol{A} = \boldsymbol{A}_1 + \boldsymbol{A}_2 j$ is such a decomposition, let

$$\boldsymbol{\chi}_A = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 \\ -\bar{\boldsymbol{A}}_2 & \bar{\boldsymbol{A}}_1 \end{bmatrix}$$

We call $\boldsymbol{\chi}_A$ the complex adjoint matrix or adjoint matrix of $\boldsymbol{A}$. It can be verified that adjoint matrices have the following properties[4]: for any $n \times n$ quaternion matrices $\boldsymbol{A}, \boldsymbol{B}$,

1) $\boldsymbol{\chi}_{AB} = \boldsymbol{\chi}_A \boldsymbol{\chi}_B$;

2) $\boldsymbol{\chi}_{A^*} = \boldsymbol{\chi}_A^*$;

3) $\boldsymbol{A}$ is unitary or Hermitian if and only if $\boldsymbol{\chi}_A$ is unitary or Hermitian respectively;

4) A Hermitian matrix $\boldsymbol{A}$ is positive definite (semi-positive definite) if and only if its standard eigenvalues are all positive (nonnegative, respectively).

2 Proof of the Main Theorem

We'll need some preliminary results given in the following.

Lemma 1 Let $\boldsymbol{A}, \boldsymbol{B}$ be two $n \times n$ Hermitian quaternion matrices. If one of them is positive definite, then there exists an invertible matrix $\boldsymbol{C}$ such that $\boldsymbol{C}^* \boldsymbol{A}\boldsymbol{C}, \boldsymbol{C}^* \boldsymbol{B}\boldsymbol{C}$ are all diagonal.

Proof It is similar to the proof in the case of complex matrices.

Lemma 2 Let $\boldsymbol{A}, \boldsymbol{B}$ be two $n \times n$ Hermitian quaternion matrices. Let k be any fixed real number. Then $\boldsymbol{A}, \boldsymbol{B}$ can be simultaneously diagonalized by congruence if and only if $\boldsymbol{A}$, $k\boldsymbol{A} + \boldsymbol{B}$ can be simultaneously diagonalized by congruence.

Proof It is clear.

Lemma 3 Let $\boldsymbol{K}$ be the Hermitian complex matrix

$$\begin{bmatrix} d_1 & 0 & \cdots & 0 & v_1 \\ 0 & d_2 & \cdots & 0 & v_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & d_r & v_r \\ \bar{v}_1 & \bar{v}_2 & \cdots & \bar{v}_r & d_{r+1} \end{bmatrix}$$

If $\boldsymbol{K}$ is semi-positive definite and $d_{r+1} \neq 0$, then there exists a sufficiently large real number k such that $k\boldsymbol{T} + \boldsymbol{K}$ is positive definite, where

$$\boldsymbol{T} = \mathrm{diag}\{1, 1, \cdots, 1, 0\}$$

Proof Let k be sufficiently large such that $k + d_1, k + d_2, \cdots, k + d_r$ are positive. It can be easily obtained that the determinant of $k\boldsymbol{T} + \boldsymbol{K}$ is equal to

$$(k + d_1)(k + d_2)\cdots(k + d_r) \cdot \left(d_{r+1} - \frac{v_1 \bar{v}_1}{k + d_1} - \cdots - \frac{v_r \bar{v}_r}{k + d_r} \right)$$

Since $d_{r+1} > 0$, it is clear that $\det(k\boldsymbol{T} + \boldsymbol{K}) > 0$ when k is sufficiently large, and then $k\boldsymbol{T} + \boldsymbol{K}$ is positive definite.

Lemma 4 For each quaternion matrix $\boldsymbol{A}$, $r(\chi_{A}) = 2r(\boldsymbol{A})$, where χ_A is the adjoint matrix of $\boldsymbol{A}$.

Proof See theorem 7.3 of Ref.[4].

Lemma 5 For each Hermitian quaternion matrix $\boldsymbol{A}$, χ_A is positive definite (semi-positive definite) matrix as a complex matrix if and only if $\boldsymbol{A}$ is positive definite (semi-positive definite) as a quaternion matrix.

Proof By Ref.[4], the spectrum of χ_A consists of the standard eigenvalues and their conjugates. So the lemma follows.

Proof of Theorem We proceed the proof by induction on $r = \max\{r(\boldsymbol{A}), r(\boldsymbol{B})\}$, where $r(\boldsymbol{A})$, $r(\boldsymbol{B})$ denote the rank of $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively. If $r = n$, then the conclusion follows from lemma 1. Suppose that the conclusion holds for the case $\max\{r(\boldsymbol{A}), r(\boldsymbol{B})\} > r$. Now we consider the case $\max\{r(\boldsymbol{A}), r(\boldsymbol{B})\} = r$. No loss of generalities, suppose $r(\boldsymbol{A}) = r$ and $\boldsymbol{A} = \operatorname{diag}\{\underbrace{1,1,\cdots,1}_{r},0,\cdots,0\}$. Let

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{B}_{11} & \boldsymbol{B}_{12} \\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22} \end{bmatrix}$$

where $\boldsymbol{B}_{11}, \boldsymbol{B}_{12}, \boldsymbol{B}_{21}, \boldsymbol{B}_{22}$ are $r \times r$, $r \times (n-r)$, $(n-r) \times r$, $(n-r) \times (n-r)$ matrices, respectively. Since $\boldsymbol{B}_{11}$ is also semi-positive definite, there exists a unitary matrix $\boldsymbol{P}$ such that $\boldsymbol{P}^* \boldsymbol{B}_{11} \boldsymbol{P}$ is a diagonal matrix.

Suppose $\boldsymbol{P}^* \boldsymbol{B}_{11} \boldsymbol{P} = \operatorname{diag}\{d_1, d_2, \cdots, d_r\}$. If we let

$$\boldsymbol{U} = \begin{bmatrix} \boldsymbol{P} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{I} \end{bmatrix}$$

then

$$\boldsymbol{U}^* \boldsymbol{A} \boldsymbol{U} = \boldsymbol{A}$$

$$\boldsymbol{U}^* \boldsymbol{B} \boldsymbol{U} = \begin{bmatrix} \boldsymbol{P}^* \boldsymbol{B}_{11} \boldsymbol{P} & \widetilde{\boldsymbol{B}}_{12} \\ \widetilde{\boldsymbol{B}}_{21} & \widetilde{\boldsymbol{B}}_{22} \end{bmatrix}$$

So, we may omit the tilde and suppose at the beginning that $\boldsymbol{B}_{11}$ is diagonal:

$$\boldsymbol{B}_{11} = \operatorname{diag}\{d_1, d_2, \cdots, d_r\}$$

If $\boldsymbol{B}_{12} = \boldsymbol{O}$, $\boldsymbol{B}_{21} = \boldsymbol{O}$, then, since $\boldsymbol{B}_{22}$ is Hermitian, we can easily find an invertible matrix $\boldsymbol{C}$, such that $\boldsymbol{C}^* \boldsymbol{A} \boldsymbol{C}, \boldsymbol{C}^* \boldsymbol{B} \boldsymbol{C}$ are all diagonal.

Suppose $\boldsymbol{B}_{12} \neq \boldsymbol{O}$. If the first column of $\boldsymbol{B}_{12}$ is zero, then there exists some permutation matrix $\boldsymbol{P}$ such that

$$\boldsymbol{P}^* \boldsymbol{A} \boldsymbol{P} = \boldsymbol{A},\ \boldsymbol{P}^* \boldsymbol{B} \boldsymbol{P} = \begin{bmatrix} \boldsymbol{B}_{11} & \widetilde{\boldsymbol{B}}_{12} \\ \widetilde{\boldsymbol{B}}_{21} & \widetilde{\boldsymbol{B}}_{22} \end{bmatrix}$$

and the first column of $\widetilde{\boldsymbol{B}}_{12}$ isn't zero. So, as in the previous discussion, we may omit the tilde and suppose at the beginning that the first column of $\boldsymbol{B}_{12}$ isn't zero. Hence, we can rewrite $\boldsymbol{B}$ as

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{K}_{11} & \boldsymbol{K}_{12} \\ \boldsymbol{K}_{21} & \boldsymbol{K}_{22} \end{bmatrix}$$

where

$$\boldsymbol{K}_{11} = \begin{bmatrix} d_1 & 0 & \cdots & 0 & v_1 \\ 0 & d_2 & \cdots & 0 & v_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & d_r & v_r \\ \bar{v}_1 & \bar{v}_2 & \cdots & \bar{v}_r & d_{r+1} \end{bmatrix}$$

such that v_1, v_2, $\cdots$, v_r are not all zero, $\boldsymbol{K}_{12}, \boldsymbol{K}_{21}, \boldsymbol{K}_{22}$ are some $(r+1) \times (n-r-1)$, $(n-r-1) \times (r+1)$, $(n-r-1) \times (n-r-1)$ matrices, respectively. Since $\boldsymbol{B}$ is semi-positive definite, so $d_1, d_2, \cdots, d_{r+1}$ are non-negative and $d_{r+1} \neq 0$.

Let k be a positive real number. For estimating the rank of $k\boldsymbol{A} + \boldsymbol{B}$, we consider its adjoint matrix χ_{kA+B}. By lemma 5, χ_{kA+B} is semi-positive definite as a complex matrix. It is not difficult to see that its principal submatrix of order $2r+1$ composed by the elements of the first $r+1$ rows, $(n+1)$-th, $(n+2)$-th, $\cdots$, $(n+r)$-th rows and the respective columns are

$$\begin{bmatrix} \boldsymbol{K}_1 & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{K}_2 \end{bmatrix}$$

where

$$\boldsymbol{K}_1 = \begin{bmatrix} k+d_1 & 0 & \cdots & 0 & \mu_1 \\ 0 & k+d_2 & \cdots & 0 & \mu_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & k+d_r & \mu_r \\ \bar{\mu}_1 & \bar{\mu}_2 & \cdots & \bar{\mu}_r & d_{r+1} \end{bmatrix}$$

and

$$\boldsymbol{K}_2 = \begin{bmatrix} k+d_1 & 0 & \cdots & 0 \\ 0 & k+d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & k+d_r \end{bmatrix}$$

So, by the proof of lemma 3, when $k > 0$ is sufficiently large, the $(r+1) \times (r+1)$ submatrix of the upper-left hand of this matrix is positive definite, and therefore, this matrix is positive definite. In particular, $r(\chi_{kA+B}) > 2r$. Hence, by lemma 4, the rank of semi-positive definite quaternion matrix $k\boldsymbol{A} +$

$\boldsymbol{B}$ is not less than $r+1$. By the induction, $\boldsymbol{A}$ and $k\boldsymbol{A}+\boldsymbol{B}$ can be simultaneously diagonalized by congruence. Hence, the theorem follows from lemma 2.

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2个四元数矩阵的同时对角化问题

周建华

(东南大学数学系,南京 210096)

摘　要　讨论四元数 Hermitian 矩阵对在共轭合同关系下的同时对角化问题.利用与每个四元数矩阵相关联的复伴随矩阵,问题被简化为关于复数矩阵的并行问题.证明了任意2个半正定四元数矩阵在共轭合同关系下均可同时对角化.

关键词　半正定矩阵;四元数矩阵;伴随矩阵;合同

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