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Paired bialgebras and braided bialgebras

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Abstract: First, we present semisimple properties of twisted products by means of constructing an algebra isomorphism between twisted products and crossed products, and point out that there exist some relations among braided bialgebras, paired bialgebras and Yang-Baxter coalgebras. Furthermore, we give an example to illustrate these relations by using Sweedler's 4-dimensional Hopf algebra. Finally, from starting off with Yang-Baxter coalgebras, we can construct some quadratic bialgebras such that they are braided bialgebras.

Key words: braided bialgebras; paired bialgebras; twisted products; quadratic bialgebras

Twisted products and braided bialgebras have close connections with solutions of quantum-Yang-Baxter equations and this problem has been of concern to many mathematicians and physicists, thus it is of vital significance to study them.

This paper has two sections. In the first section, we present the semisimplicity and the semiprimality of twisted products by means of constructing an algebra isomorphism between twisted products and crossed products. In the second section, we show that braided bialgebras are closely related to paired bialgebras and discuss relations between paired bialgebras and braided bialgebras by using Sweedler's 4-dimensional Hopf algebra. Finally, we construct some quadratic bialgebras using Yang-Baxter coalgebras.

1 Twisted Products and Crossed Products

We always work over a fixed field k and follow Montgomery's book for terminology on coalgebras, bialgebras and Hopf algebras.

Suppose that H and K are two bialgebras and $\sigma: H \times K \to k$ is a bilinear map such that the following conditions hold:

① $\sigma(h, 1) = \varepsilon_H(h)$;

② $\sigma(1, g) = \varepsilon_H(g)$;

③ $\sigma(hg, x) = \sum \sigma(h, x_1)\sigma(g, x_2)$;

④ $\sigma(x, hg) = \sum \sigma(x_1, h)\sigma(x_2, g)$.

Then we call $[H, K, \sigma]$ paired bialgebras, see Refs. [1,2].

If H is a Hopf algebra with an antipode S_H, then σ is convolution invertible, the inverse of which is given by $\sigma^{-1}(h, k) = \sigma(S_H(h), k)$.

Let $[H, K, \sigma]$ be paired bialgebras, A be an (H, K)-bicomodule algebra. Define $a *_\sigma b = \sum \sigma(a_{-1}, b_1)a_0 b_0$, then $(A, *_\sigma)$ is an associative algebra and is denoted by A^σ, which is called the twisted product. Here the left H-comodule structure map and right K-comodule structure map of A are denoted by $\rho_{A^-}(a) = \sum a_{-1} \otimes a_0 \in H \otimes A$, $\rho_{A^+}(a) = \sum a_0 \otimes a_1 \in A \otimes K$, respectively.

Let H be a Hopf algebra and A be an algebra. Assuming that H weakly acts on A and $\sigma: H \times H \to A$ is a k-linear map. The crossed product in Ref. [3] $A \#_\sigma H$ is the tensor product $A \otimes H$ as a vector space, the multiplication of which is given by

$$(a \#_\sigma h)(b \#_\sigma g) = \sum a(h_1 \cdot b)\sigma(h_2, g_1) \#_\sigma h_3 g_2 \qquad \text{for all } h, g \in H;\ a, b \in A$$

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If $A\#_\sigma H$ is an associative algebra with an identity element $1\#_\sigma 1$, then it is called a crossed product algebra.

Lemma 1[4] $A\#_\sigma H$ is a crossed product algebra if and only if the following conditions hold:

① A is a twisted H-module, that is, for all h, $k\in H$, $a\in A$, $1\cdot a=a$, and

$$h\cdot(k\cdot a)=\sum\sigma(h_1,k_1)(h_2k_2\cdot a)\sigma^{-1}(h_3,k_3)\tag{1}$$

② σ is a cocycle. That is, for all g, h, $k\in H$, $\sigma(h,1)=\varepsilon_H(h)1_A=\sigma(1,h)$, and

$$\sum[h_1\cdot\sigma(k_1,g_1)]\sigma(h_2,k_2g_2)=\sum\sigma(h_1,k_1)\sigma(h_2k_2,g)\tag{2}$$

Let $[H,H,\sigma]$ be paired bialgebras. Then H is an (H,H)-bicomodule algebra via the comultiplication Δ_H: $H\to H\otimes H$ and $(H^\sigma,*)$ is a twisted product, the multiplication of which is given by $g*h=\sum\sigma(g_1,h_1)g_2h_2$.

Proposition 1 Let H be a Hopf algebra, $[H,H,\sigma]$ be paired bialgebras. Then we can conclude the following:

① If $k\#_\sigma H$ is a crossed product algebra, then there exists an algebra isomorphism: $H^\sigma\cong k\#_\sigma H$. In particular, when H is cocommutative, $k\#_\sigma H$ is a crossed product algebra.

② If H is cocommutative, then $\sigma^2=\sigma$ if and only if Δ_H is a multiplication map in H^σ. Here $\sigma^2=\sigma*\sigma$, "$*$" denotes the convolution product multiplication and the comultiplication in H^σ is exactly Δ_H.

③ σ is trivial, that is, $\sigma=\varepsilon_H\otimes\varepsilon_H$, if and only if ε_H is a multiplicative map in H^σ.

④ If H^σ is commutative, then σ is commutative, that is, for all g, $h\in H$, $\sigma(g,h)=\sigma(h,g)$.

Proof ① Let F: $H^\sigma\to k\#_\sigma H$, $h\mapsto 1\#_\sigma h$. Then it is easy to prove that F is an algebra isomorphism.

② If H is cocommutative, then for all g, $h\in H^\sigma$, we have

$$\Delta_H(g*h)=\sum\Delta_H(\sigma(g_1,h_1)g_2h_2)=\sum\sigma(g_1,h_1)g_2h_2\otimes g_3h_3$$

$$\Delta_H(g)*\Delta_H(h)=\sum g_1*h_1\otimes g_2*h_2=\sum\sigma(g_1,h_1)g_2h_2\otimes\sigma(g_3,h_3)g_4h_4=$$

$$\sum\sigma(g_1,h_1)\sigma(g_2,h_2)g_3h_3\otimes g_4h_4$$

so $\sigma^2=\sigma$ if and only if $\Delta_H(g*h)=\Delta_H(g)*\Delta_H(h)$.

③ and ④ can be proved straightforwardly.

By proposition 1, if σ is not trivial, then H^σ is not a bialgebra.

Lemma 2[3] Let H be a finite-dimensional semisimple Hopf algebra and $A\#_\sigma H$ be a crossed product algebra. If σ is invertible, then we conclude the following:

① (Maschke Theorem) If A is semisimple, then $A\#_\sigma H$ is also semisimple;

② Assume the action of H on A is inner, then when A is semiprime $A\#_\sigma H$ is semiprime too.

It is obvious that the field k is both semisimple and semiprime and the action of H on k is inner. Since H is a Hopf algebra, σ is convolution invertible. Thus by proposition 1 and lemma 2, we have:

Corollary Let $k\#_\sigma H$ be a crossed product algebra and $[H,H,\sigma]$ be paired bialgebras. If H be a finite-dimensional semisimple Hopf algebra, then the twisted product H^σ is both semisimple and semiprime.

Example 1 Let $G=\{1,e\}$ be a binary cyclic group and $H=kG$ is a group algebra. Then H is a cocommutative Hopf algebra. It is easy to check that the antipode $S_H=I_H$.

Let $\sigma(1,e)=\sigma(e,1)=1,\sigma(1,1)=1,\sigma(e,e)=-1$. Then $[H,H,\sigma]$ are paired bialgebras and σ is convolution invertible with the inverse $\sigma^{-1}=\sigma$. Thus by proposition 1 we know that $k\#_\sigma H$ is a crossed product algebra.

Let Δ_H: $H\to H\otimes H$ be the comultiplication of H. Then H is an (H,H)-bicomodule algebra via Δ_H, so $(H^\sigma,*)$ is a twisted product, of which comultiplication is given by $e*1=1*e=e$, $1*1=1$, $e*e=-1$.

Suppose $t=e+1$, then $t\in\int_H$ (the set of left integrals of H) because for all $h\in H$, $ht=\varepsilon_H(h)t$. If char$k\neq 2$, then $\varepsilon_H(t)=1+1\neq 0$, by Ref. [3] we know that H is a semisimple Hopf algebra. Thus H^σ is both semisimple and semiprime by corollary.

But H^σ is not a bialgebra since σ is not trivial.

2 Paired Bialgebras and Braided Bialgebras

Let H be a bialgebra (Hopf algebra), $\sigma\in\mathrm{Hom}_k(H\otimes H,k)$ convolution invertible. (H,σ) is called a braided bialgebra (Hopf algebra)[5] if the following conditions hold:

For all x, y, $z \in H$,

① $\sum \sigma(x_1, y_1)x_2 y_2 = \sum y_1 x_1 \sigma(x_2, y_2)$, that is, (H,σ) is almost commutative;

② $\sigma(xy, z) = \sum \sigma(x, z_1)\sigma(y, z_2)$;

③ $\sigma(x, yz) = \sum \sigma(x_1, z)\sigma(x_2, y)$.

By the above conditions, we easily get

④ $\sigma(1, x) = \varepsilon_H(x) = \sigma(x, 1)$;

⑤ $\sum \sigma(x_1, y_1)\sigma(x_2 y_2, z) = \sum \sigma(y_1, z_1)\sigma(x, y_2 z_2)$;

By ②, ③ and ⑤ we get

⑥ $\sum \sigma(x_1, y_1)\sigma(x_2, z_1)\sigma(y_2, z_2) = \sum \sigma(y_1, z_1)\sigma(x_1, z_2)\sigma(x_2, y_2)$.

Let C be a coalgebra and $\sigma \in \mathrm{Hom}_k(C \otimes C, k)$ convolution invertible. If ⑥ holds, then (C,σ) is called a Yang-Baxter coalgebra[5].

Let (C,σ) be a Yang-Baxter coalgebra. Then by theorem 2.6 in Ref. [5] we may construct Doi's quadratic bialgebra $M(C,\sigma)$ such that it is a braided bialgebra.

In this section, we will mainly give some relations between braided bialgebras and the paired bialgebras.

Proposition 2 Assume that (H,σ) is a braided Hopf algebra and one of the following conditions holds:

① H is commutative;

② H is weak commutative, that is, for all $x, y \in H, \sigma(-, xy) = \sigma(-, yx)$;

③ σ is commutative, that is, for all $x, y \in H, \sigma(x, y) = \sigma(y, x)$.

Then $[H,\ H,\ \delta = \sigma T]$ are paired bialgebras and $k \#_\delta H$ is a crossed product algebra, so $k \#_\delta H \cong H^\delta$.

Here H^δ denotes the twisted product of which multiplication is given by $g * h = \sum \delta(g_1, h_1)g_2 h_2$ and T denotes twisted map.

Proof Suppose that ③ holds, then ② holds. If ② is satisfied the conclusion, then we easily prove that ① is satisfied the conclusion. Thus we have only to show that ③ is satisfied the conclusion.

Suppose that ③ is satisfied. It is easy to prove that $[H,\ H,\ \delta = \sigma T]$ are paired bialgebras. By proposition 1, we have only to prove that $k \#_\delta H$ is a crossed product algebra.

By lemma 1, we need to prove that δ satisfies the cocycle condition ②.

As a matter of fact, for all g, $h, k \in H$, we get

$$\sum \delta(k_1, g_1)\delta(h, k_2 g_2) = \sum \sigma(g_1, k_1)\sigma(k_2 g_2, h) = \sum \sigma(g_1, k_1)\sigma(g_2 k_2, h) =$$
$$\sum \sigma(\sigma(g_1, k_1)g_2 k_2, h) = \sum \sigma(k_1 g_1 \sigma(g_2, k_2), h) = \sum \sigma(k_1, h_1)\sigma(g_1, h_2)\sigma(g_2, k_2) =$$
$$\sum \sigma(k_1, h_1)\sigma(g, k_2 h_2) = \sum \sigma(k_1, h_1)\sigma(g, h_2 k_2) = \sum \delta(h_1, k_1)\delta(h_2 k_2, g)$$

that is, δ satisfies the cocycle condition. This is completed.

Example 2 Let H_4 be Sweedler's 4-dimensional Hopf algebra and $\mathrm{char}k \neq 2$, the smallest non-commutative and non-cocommutative Hopf algebra. It is described as

$$H_4 = k\langle 1,\ x,\ y,\ z \mid x^2 = 1, y^2 = 0, xy = z, xz = -zx = y\rangle$$

is a free k-module with the bases $\{1,\ x,\ y,\ z\}$, of which coalgebra structure and the antipode are given by

$$\Delta(x) = x \otimes x,\ \Delta(y) = y \otimes x + 1 \otimes y,\ \Delta(z) = z \otimes 1 + x \otimes z$$
$$\varepsilon(x) = 1,\ \varepsilon(y) = \varepsilon(z) = 0,\ S(x) = x,\ S(y) = z,\ S(z) = -y$$

For all $\alpha \in k$, define $\sigma_\alpha : H_4 \otimes H_4 \to k$ as

$$\sigma_\alpha(1,1) = \sigma_\alpha(1,x) = \sigma_\alpha(x,1) = 1,\ \sigma_\alpha(x,x) = -1$$
$$\sigma_\alpha(y,y) = \sigma_\alpha(z,y) = \sigma_\alpha(z,z) = \alpha,\ \sigma_\alpha(y,z) = -\alpha$$
$$\sigma_\alpha(1,y) = \sigma_\alpha(1,z) = \sigma_\alpha(y,1) = \sigma_\alpha(z,1) = 0$$
$$\sigma_\alpha(x,y) = \sigma_\alpha(x,z) = \sigma_\alpha(y,x) = \sigma_\alpha(z,x) = 0$$

then (H_4, σ_α) is a braided Hopf algebra. It is easy to check that (H_4, σ_0) is a non-commutative, non-cocommutative braided Hopf algebra and the condition ③ in proposition 2 holds. Thus the conclusion in proposition 2 holds.

When $\alpha \neq 0$, (H_4, σ_α) is non-commutative, non-cocommutative braided Hopf algebra, σ_α is not commutative since $\sigma_\alpha(y,z) \neq \sigma_\alpha(z,y)$.

It is obvious that H is not weak commutative, that is, the condition ② in proposition 2 is not satisfied. Furthermore, the conclusion in proposition 2 doesn't hold, this is because that

$$\delta = \sigma T, \delta(xy,z) = \delta(z,z) = \sigma_\alpha(z,z) = \alpha$$

$$\sum \delta(x,z_1)\delta(y,z_2) = \sum \sigma_\alpha(z_1,x)\sigma_\alpha(z_2,y) = \sigma_\alpha(z,x)\sigma_\alpha(1,y) + \sigma_\alpha(x,x)\sigma_\alpha(z,y) = -\alpha$$

so $[H,H,\delta]$ is not paired bialgebras. This example demonstrates that the condition ③ in proposition 2 is necessary.

Proposition 3 Let $[H,H,\sigma]$ be paired bialgebras and σ convolution invertible. Then we have the following conclusions:

① If (H,σ) is almost commutative, that is, $\sum \sigma(x_1,y_1)x_2y_2 = \sum y_1x_1\sigma(x_2,y_2)$, then (H,σ) is a Yang-Baxter coalgebra. Conversely, if σ is nonsingular and (H,σ) is a Yang-Baxter coalgebra, then (H, σ^{-1}) is almost commutative.

② Let (H,σ) be almost commutative. Then H is weak commutative, that is, for all $x,y \in H, \sigma(-,xy) = \sigma(-,yx)$ if and only if (H,σ) is a braided bialgebra.

③ If (H,σ) is almost commutative, then $[H^{op}, H^{op}, \sigma^{-1}]$ are paired bialgebras.

Proof ① Suppose that (H,σ) is almost commutative, then for all $x,y,z \in H$, we get

$$\sum \sigma(x_1,y_1)\sigma(x_2,z_1)\sigma(y_2,z_2) = \sum \sigma(x_1,y_1)\sigma(x_2y_2,z) =$$

$$\sum \sigma(\sigma(x_1,y_1)x_2y_2,z) = \sum \sigma(y_1x_1\sigma(x_2,y_2),z) = \sum \sigma(y_1,z_1)\sigma(x_1,z_2)\sigma(x_2,y_2)$$

and (H,σ) is a Yang-Baxter coalgebra.

Conversely, suppose that (H,σ) is a Yang-Baxter coalgebra, since $[H,H,\sigma]$ are paired bialgebras, we get

$$\sum \sigma(x,y_1z_1)\sigma(y_2,z_2) = \sum \sigma(y_1,z_1)\sigma(x,z_2y_2)$$

It follows from σ being nonsingular that $\sum y_1z_1\sigma(y_2,z_2) = \sum \sigma(y_1,z_1)z_2y_2$. This equality implies $\sum \sigma^{-1}(y_1,z_1)y_2z_2 = \sum z_1y_1\sigma^{-1}(y_2,z_2)$, that is, (H,σ^{-1}) is almost commutative.

② can be proved straightforwardly.

③ Let (H,σ) be almost commutative. Then one easily checks $\sum \sigma^{-1}(x_1,y_1)y_2x_2 = \sum x_1y_1\sigma^{-1}(x_2,y_2)$ and (H^{op},σ^{-1}) is almost commutative. Thus $[H^{op},H^{op},\sigma^{-1}]$ are paired bialgebras by the proof of proposition 1.3 in Ref.[2]. This is because

$$\sigma^{-1}(x,y\circ z) = \sigma^{-1}(x,zy) = \sum \sigma^{-1}(x_1,y)\sigma^{-1}(x_2,z)$$

$$\sigma^{-1}(x\circ y,z) = \sigma^{-1}(yx,z) = \sum \sigma^{-1}(y,z_2)\sigma^{-1}(x,z_1) = \sum \sigma^{-1}(x,z_1)\sigma^{-1}(y,z_2)$$

$$\sigma^{-1}(x,1) = \varepsilon_H(x),\ \sigma^{-1}(1,y) = \varepsilon_H(y)$$

This proof is completed.

Remark The concept of braided Hopf algebras is dual to the concept of quasitriangular Hopf algebras, and the concept of paired bialgebras is dual to the concept of copaired bialgebras, thus we may study some dual properties on quasitriangular Hopf algebras and copaired bialgebras.

Finally, we will construct Doi's quadratic bialgebras by Yang-Baxter coalgebras.

Lemma 3 Let (C,σ) be a Yang-Baxter coalgebra and σ covolution invertible. Then (C,σ^{-1}) is also a Yang-Baxter coalgebra.

Proof For all $x,y,z \in C$, by ⑥ we can get

$$\sum \sigma^{-1}(y_1,z_1)\sigma^{-1}(x_1,z_2)\sigma^{-1}(x_2,y_2) =$$

$$\sum \sigma^{-1}(x_1,y_1)\sigma^{-1}(x_2,z_1)\sigma^{-1}(y_2,z_2)\sigma(y_3,z_3)\sigma(x_3,z_4)\sigma(x_4,y_4)\sigma^{-1}(y_5,z_5)\sigma^{-1}(x_5,z_6)\sigma^{-1}(x_6,y_6) =$$

$$\sum \sigma^{-1}(x_1,y_1)\sigma^{-1}(x_2,z_1)\sigma^{-1}(y_2,z_2)\sigma(x_3,y_3)\sigma(x_4,z_3)\sigma(y_4,z_4)\sigma^{-1}(y_5,z_5)\sigma^{-1}(x_5,z_6)\sigma^{-1}(x_6,y_6) =$$

$$\sum \sigma^{-1}(x_1,y_1)\sigma^{-1}(x_2,z_1)\sigma^{-1}(y_2,z_2)\sigma(x_3,y_3)\sigma(x_4,z_3)\sigma^{-1}(x_5,z_4)\sigma^{-1}(x_6,y_4) =$$

$$\sum \sigma^{-1}(x_1,y_1)\sigma^{-1}(x_2,z_1)\sigma^{-1}(y_2,z_2)\sigma(x_3,y_3)\sigma^{-1}(x_4,y_4) =$$

$$\sum \sigma^{-1}(x_1, y_1)\sigma^{-1}(x_2, z_1)\sigma^{-1}(y_2, z_2)$$

Then (C, σ^{-1}) is a Yang-Baxter coalgebra.

Assuming that (C, σ) is a Yang-Baxter coalgebra, it is easy to show that $(C^{\mathrm{cop}}, \sigma)$ is also a Yang-Baxter coalgebra. Thus by lemma 3 and theorem 2.6 in Ref.[5] we get

Proposition 4 Let (C, σ) be a Yang-Baxter coalgebra and σ covolution invertible. Then $M(C^{\mathrm{cop}}, \sigma)$, $M(C, \sigma^{-1})$ and $M(C^{\mathrm{cop}}, \sigma^{-1})$ are quadratic bialgebras and braided bialgebras.

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对偶双代数和辫化双代数

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摘　要　通过建立扭曲积和交叉积之间的代数同构, 首先得到了扭曲积的半单性质. 指出了对偶双代数、Yang-Baxter 余代数和辫化双代数之间的关系, 并且以四维 Sweedler Hopf 代数为例来说明. 最后由 Yang-Baxter 余代数出发, 构造二次双代数使之成为辫化双代数.

关键词　辫化双代数; 对偶双代数; 扭曲积; 二次双代数

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