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Stability analysis for nonlinear multi-variable delay perturbation problems

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Abstract: This paper discusses the stability of theoretical solutions for nonlinear multi-variable delay perturbation problems (MVDPP) of the form $x'(t) = f(x(t), x(t-\tau_1(t)), \cdots, x(t-\tau_m(t)), y(t), y(t-\tau_1(t)), \cdots, y(t-\tau_m(t)))$, and $\varepsilon y'(t) = g(x(t), x(t-\tau_1(t)), \cdots, x(t-\tau_m(t)), y(t), y(t-\tau_1(t)), \cdots, y(t-\tau_m(t)))$, where $0 < \varepsilon \ll 1$. A sufficient condition of stability for the systems is obtained. Additionally we prove the numerical solutions of the implicit Euler method are stable under this condition.

Key words: multi-variable delay perturbation problems; Euler method; stability; interpolation

Multi-variable delay perturbation problems (MVDPP) usually arise in many hi-tech fields such as automatic control, electronic systems, biology, etc. They are deemed as a sub-system of delay differential equations (DDEs). There are many results concerning the numerical stability and convergence for DDEs[1-4]. However, little research work concerning the stability of singular perturbation problems with delays (SPPDs) especially MVDPP has been reported at home and abroad. Consider the following systems:

Let $\langle \cdot, \cdot \rangle$ be the inner product of Euclidean space with the norm $\|\cdot\|$.

$$\left.\begin{aligned} &x'(t) = f(x(t), x(t-\tau_1(t)), \cdots, x(t-\tau_m(t)), y(t), y(t-\tau_1(t)), \cdots, y(t-\tau_m(t))) \\ &\qquad t \in [0, T] \\ &\varepsilon y'(t) = g(x(t), x(t-\tau_1(t)), \cdots, x(t-\tau_m(t)), y(t), y(t-\tau_1(t)), \cdots, y(t-\tau_m(t))) \\ &\qquad t \in [0, T]; 0 < \varepsilon \ll 1 \\ &x(t) = \phi(t),\ y(t) = \varphi(t) \qquad t^* \leqslant t \leqslant 0 \end{aligned}\right\} \tag{1}$$

$$\left.\begin{aligned} &w'(t) = f(w(t), w(t-\tau_1(t)), \cdots, w(t-\tau_m(t)), z(t), z(t-\tau_1(t)), \cdots, z(t-\tau_m(t))) \\ &\qquad t \in [0, T] \\ &\varepsilon z'(t) = g(w(t), w(t-\tau_1(t)), \cdots, w(t-\tau_m(t)), z(t), z(t-\tau_1(t)), \cdots, z(t-\tau_m(t))) \\ &\qquad t \in [0, T]; 0 < \varepsilon \ll 1 \\ &x(t) = \theta(t), y(t) = \vartheta(t) \qquad t^* \leqslant t \leqslant 0 \end{aligned}\right\} \tag{2}$$

where ε is a constant; θ, ϑ, ϕ and φ denote the given continuous functions; $f: C^M \times \underbrace{C^M \times \cdots \times C^M}_{m} \times C^N \times \underbrace{C^N \times \cdots \times C^N}_{m} \to C^M$ and $g: C^M \times \underbrace{C^M \times \cdots \times C^M}_{m} \times C^N \times \underbrace{C^N \times \cdots \times C^N}_{m} \to C^N$ are the given sufficient smooth mappings; $t^* = \inf\limits_{t \geqslant t_0, 1 \leqslant i \leqslant m} \{t - \tau_i(t)\}, \tau_i(t) > 0\ (i = 1, 2, \cdots, m)$. It is always assumed that (1) has a unique solution $x(t)$ and (2) has a unique solution $w(t)$. $\bar{\varepsilon} = 1 + 1/\varepsilon$.

1 Stability of Theoretical Solutions

Definition 1 System (1) is stable if the solutions $y(t)$ and $Z(t)$ corresponding to different initial functions θ, ϑ and ϕ, φ, respectively, satisfy

$$\|x(t) - w(t)\| + \|y(t) - z(t)\| \leqslant C \max_{t^* \leqslant t \leqslant t_0} \{\|\phi(t) - \theta(t)\| + \|\varphi(t) - \vartheta(t)\|\} \tag{3}$$

where C is a constant.

Theorem 1 Assume system (1) satisfies

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$$\operatorname{Re}\langle f(x_1,u_1,\cdots,u_m,y,v_1,\cdots,v_m)-f(x_2,u_1,\cdots,u_m,y,v_1,\cdots,v_m),x_1-x_2\rangle \leqslant \omega_1(t)\|x_1-x_2\|^2 \tag{4}$$

$$\operatorname{Re}\langle g(x,u_1,\cdots,u_m,y_1,v_1,\cdots,v_m)-g(x,u_1,\cdots,u_m,y_2,v_1,\cdots,v_m),y_1-y_2\rangle \leqslant \omega_2(t)\|y_1-y_2\|^2 \tag{5}$$

$$\begin{aligned}&\|f(x_0,x_1,\cdots,x_m,y_0,y_1,\cdots,y_m)-f(\bar{x}_0,\bar{x}_1,\cdots,\bar{x}_m,\bar{y}_0,\bar{y}_1,\cdots,\bar{y}_m)\|\leqslant\\&\qquad\sum_{j=0}^{m}\sigma_j(t)\|x_j-\bar{x}_j\|+\sum_{j=0}^{m}\gamma_j(t)\|y_j-\bar{y}_j\|\end{aligned} \tag{6}$$

$$\begin{aligned}&\|g(x_0,x_1,\cdots,x_m,y_0,y_1,\cdots,y_m)-g(\bar{x}_0,\bar{x}_1,\cdots,\bar{x}_m,\bar{y}_0,\bar{y}_1,\cdots,\bar{y}_m)\|\leqslant\\&\qquad\sum_{j=0}^{m}\bar{\sigma}_j(t)\|x_j-\bar{x}_j\|+\sum_{j=0}^{m}\bar{\gamma}_j(t)\|y_j-\bar{y}_j\|\end{aligned} \tag{7}$$

then system (1) is stable.

Here $\omega_1(t),\omega_2(t),\sigma_j(t),\bar{\sigma}_j(t),\gamma_j(t),\bar{\gamma}_j(t)(j=1,2,\cdots,m)$ are continuous functions; $\sigma(t)=\sum_{j=1}^{m}\sigma_j(t)$, $\gamma(t)=\sum_{j=1}^{m}\gamma_j(t)$, $\bar{\sigma}(t)=\sum_{j=1}^{m}\bar{\sigma}_j(t)$, $\bar{\gamma}=\sum_{j=1}^{m}\bar{\gamma}_j(t)$. And they satisfy $\bar{\sigma}_0(t)$, $\gamma_0(t)<0$, $\sigma(t)>0$, $\bar{\sigma}(t)\leqslant-\omega_1(t)$; $\gamma(t)>0$, $\bar{\gamma}(t)\leqslant-\omega_2(t)$.

Proof Let $X(t)=\|x(t)-w(t)\|,Y(t)=\|y(t)-z(t)\|,E=\{t\in[t_0,\infty):\|Y(t)\text{ or }X(t)=0\|\}$.

When $t\in E$, system (1) is stable. This is readily seen[5]. When $t\in[t_0,\infty)/E$, then $Y'(t)$ and $X'(t)$ exit. And

$$X'(t)=\frac{\operatorname{Re}\langle x'(t)-w'(t),x(t)-w(t)\rangle}{X(t)},Y'(t)=\frac{\operatorname{Re}\langle y'(t)-z'(t),y(t)-z(t)\rangle}{Y(t)}$$

From Schwaritz inequality, (4) and (6), it follows

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)-w(t)\|^2&=\operatorname{Re}\langle x'(t)-w'(t),x(t)-w(t)\rangle\leqslant\\&\qquad\omega_1(t)X(t)^2+\gamma_0(t)Y(t)X(t)+\Gamma(t)X(t)+\Psi(t)X(t)\end{aligned}$$

Therefore

$$X'(t)\leqslant\omega_1(t)X(t)+\gamma_0(t)Y(t)+\Gamma(t)+\Psi(t) \tag{8}$$

where $\Gamma(t)=\sum_{j=1}^{m}\sigma_j(t)\|x(t-\tau_j(t))-w(t-\tau_j(t))\|,\Psi(t)=\sum_{j=1}^{m}\gamma_j(t)\|y(t-\tau_j(t))-z(t-\tau_j(t))\|$.

Using the same method, we obtain

$$Y'(t)\leqslant\frac{1}{\varepsilon}\omega_2(t)Y(t)+\frac{1}{\varepsilon}\bar{\sigma}_0(t)X(t)+\bar{\Gamma}(t)+\bar{\Psi}(t) \tag{9}$$

where

$$\bar{\Gamma}(t)=\frac{1}{\varepsilon}\sum_{j=1}^{m}\bar{\sigma}_j(t)\|x(t-\tau_j(t))-w(t-\tau_j(t))\|$$

$$\bar{\Psi}(t)=\frac{1}{\varepsilon}\sum_{j=1}^{m}\bar{\gamma}_j(t)\|y(t-\tau_j(t))-z(t-\tau_j(t))\|$$

Let

$$G(t)=X(t)+Y(t),B(t)=\max\left\{\left(\omega_1(t)+\frac{1}{\varepsilon}\bar{\sigma}_0(t)\right),\left(\gamma_0(t)+\frac{1}{\varepsilon}\omega_2(t)\right)\right\}$$

$$\begin{aligned}\Omega(t)=&\sum_{j=1}^{m}\left(\sigma_j(t)+\frac{1}{\varepsilon}\bar{\sigma}(t)_j\right)\|x(t-\tau_j(t))-w(t-\tau_j(t))\|+\\&\sum_{j=1}^{m}\left(\gamma_j(t)+\frac{1}{\varepsilon}\bar{\gamma}_j(t)\right)\|y(t-\tau_j(t))-z(t-\tau_j(t))\|\end{aligned}$$

From (8) and (9), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(G(t)\exp(-A(t)))\leqslant\exp(-A(t))\Omega(t)$$

where $A(t)=\int_{t_0}^{t}B(x)\mathrm{d}x$.

Therefore

$$G(t)\leqslant\exp(A(t))\left((G(t_0)+\int_{t_0}^{t}\Omega(x)\exp(-A(x))\mathrm{d}x\right)$$

$$G(t) \leqslant \exp(A(t))(G(t_0) + \max_{t_0 \leqslant x \leqslant t}(-B(x))^{-1}\Omega(x)(\exp(-A(t)) - 1))$$

$$G(t) \leqslant \max(G(t_0), C_1 \max_{1 \leqslant j \leqslant m}(\|X(t-\tau_j(t))\| + \|Y(t-\tau_j(t))\|))$$

where $C_1 = \bar{\varepsilon} \max\limits_{t_0 \leqslant t \leqslant T}\left\{\dfrac{\omega_1(t)}{B(t)}, \dfrac{\omega_2(t)}{B(t)}\right\}$.

Further, for $\forall\, t \geqslant t_0$, we get

$$\|x(t) - w(t)\| + \|y(t) - z(t)\| \leqslant C \max_{t^* \leqslant x \leqslant t_0}\{\|\phi(t) - \theta(t)\| + \|\varphi(t) - \vartheta(t))\|\}$$

with

$$C = C_1^K,\ K = \left[\frac{T}{\min\{\inf\limits_{t_0 \leqslant t \leqslant T}(\tau_1(t), \cdots, \tau_m(t))\}}\right] + 1$$

where [ ] denotes the integer function.

2 The Stability of the Implicit Euler Method

When applying the implicit Euler method to system (1) and system (2)

$$\left.\begin{aligned} x_{n+1} &= x_n + hf(x_{n+1}, \bar{x}_{1,n+1}, \cdots, \bar{x}_{m,n+1}, y_{n+1}, \bar{y}_{1,n+1}, \cdots, \bar{y}_{m,n+1}) \\ y_{n+1} &= y_n + \frac{1}{\varepsilon}hg(x_{n+1}, \bar{x}_{1,n+1}, \cdots, \bar{x}_{m,n+1}, y_{n+1}, \bar{y}_{1,n+1}, \cdots, \bar{y}_{m,n+1}) \end{aligned}\right\} \tag{10}$$

$$\left.\begin{aligned} w_{n+1} &= w_n + hf(w_{n+1}, \tilde{w}_{1,n+1}, \cdots, \tilde{w}_{m,n+1}, z_{n+1}, \tilde{z}_{1,n+1}, \cdots, \tilde{z}_{m,n+1}) \\ z_{n+1} &= z_n + \frac{1}{\varepsilon}hg(w_{n+1}, \tilde{w}_{1,n+1}, \cdots, \tilde{w}_{m,n+1}, z_{n+1}, \tilde{z}_{1,n+1}, \cdots, \tilde{z}_{m,n+1}) \end{aligned}\right\} \tag{11}$$

where $x_n, y_n, w_n, z_n, \bar{x}_{j,n+1}, \bar{y}_{j,n+1}, \tilde{w}_{j,n+1}$ and $\tilde{z}_{j,n+1}$ are approximations to true solutions $x(t), y(t), w(t), z(t)$, $x(t-\tau_j(t)), y(t-\tau_j(t)), w(t-\tau_j(t))$ and $z(t-\tau_j(t))$.

Definition 2 The numerical solutions of system (1) are stable if they satisfy

$$\|x_n - w_n\| + \|y_n - z_n\| \leqslant C \max_{t^* \leqslant t \leqslant t_0}\{\|\phi(t) - \theta(t)\| + \|\varphi(t) - \vartheta(t))\|\} \qquad n \geqslant 1$$

Definition 3 Interpolation methods are stable if $\bar{x}, \bar{y}, \bar{w}$ and $\bar{z}$ satisfy

$$\|\bar{y} - \bar{z}\| + \|\bar{x} - \bar{w}\| \leqslant L_0 \max\{\max_{0 < i \leqslant n}\{\|y_i - z_i\| + \|x_i - w_i\|\}, \max_{t \leqslant 0}\{\|\phi(t) - \theta(t)\| + \|\varphi(t) - \vartheta(t))\|\}\}$$

where $\{x_n\}$, $\{y_n\}$, $\{w_n\}$, $\{z_n\}$ are respectively solution sequences of (10) and (11) and $\bar{x}, \bar{y}, \bar{w}, \bar{z}$ are the corresponding interpolations at $t = t^*(t_0 \leqslant t^* < t_n)$.

Theorem 2 The numerical solutions of system (1) are stable if the system satisfies (4) - (7) and the interpolation method is stable.

Proof Let $\Delta x_{n+1} = x_{n+1} - w_{n+1}, \Delta y_{n+1} = y_{n+1} - z_{n+1}, \Delta\bar{x}_{j,n+1} = \bar{x}_{j,n+1} - \tilde{w}_{j,n+1}$, $\Delta\tilde{y}_{n+1} = \bar{y}_{j,n+1} - \tilde{z}_{j,n+1}$, $\Theta = \varepsilon \max\limits_{0 \leqslant t \leqslant T}\{-w_1(t), -w_2(t)\}$.

$$\|\Delta x_{n+1}\|^2 = \langle \Delta x_{n+1}, \Delta x_{n+1}\rangle = \langle x_{n+1} - w_{n+1}, x_n - w_n\rangle + h(f(x_{n+1}, \bar{x}_{1,n+1}, \cdots, \bar{x}_{m,n+1}, y_{n+1}, \bar{y}_{1,n+1}, \cdots, \bar{y}_{m,n+1}) - f(w_{n+1}, \tilde{w}_{1,n+1}, \cdots, \tilde{w}_{m,n+1}, z_{n+1}, \tilde{z}_{1,n+1}, \cdots, \tilde{z}_{m,n+1}))$$

From (4) -(7), we get

$$\|\Delta x_{n+1}\| \leqslant \|\Delta x_n\| + h\sum_{j=1}^{m}\sigma_j(t)\|\Delta\bar{x}_{j,n+1}\| + h\sum_{j=1}^{m}\gamma_j(t)\|\Delta\tilde{y}_{j,n+1}\|$$

Using the same method, we have

$$\|\Delta y_{n+1}\| \leqslant \|\Delta y_n\| + \frac{h}{\varepsilon}\sum_{j=1}^{m}\bar{\sigma}_j(t)\|\Delta\bar{x}_{j,n+1}\| + \frac{h}{\varepsilon}\sum_{j=1}^{m}\bar{\gamma}_j(t)\|\Delta\tilde{y}_{j,n+1}\|$$

Therefore

$$\|\Delta x_{n+1}\| + \|\Delta y_{n+1}\| \leqslant \|\Delta x_n\| + \|\Delta y_n\| + h\bar{\varepsilon}\Theta \max_{1 \leqslant j \leqslant m}\{\|\Delta\bar{x}_{j,n+1}\| + \|\Delta\tilde{y}_{j,n+1}\|\}$$

By the interpolation method, we obtain

$$\|\Delta x_{n+1}\| + \|\Delta y_{n+1}\| \leqslant (1 + hL\Theta)\max_{1 \leqslant j \leqslant n}\{\|\Delta x_j\| + \|\Delta y_j\|, \|\phi(t) - \theta(t)\| + \|\varphi(t) - \vartheta(t)\|\}$$

Further,

$$\|x_n - w_n\| + \|y_n - z_n\| \leqslant C \max_{t^* \leqslant t \leqslant t_0}\{\|\phi(t) - \theta(t)\| + \|\varphi(t) - \vartheta t)\|\}, C = \exp(\Theta LT)$$

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非线性多变延迟奇异摄动问题的稳定性分析

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摘　要　讨论了形如 $x'(t)=f(x(t),x(t-\tau_1(t)),\cdots,x(t-\tau_m(t)),y(t),y(t-\tau_1(t)),\cdots,y(t-\tau_m(t)))$和 $\varepsilon y'(t)=g(x(t),x(t-\tau_1(t)),\cdots,x(t-\tau_m(t)),y(t),y(t-\tau_1(t)),\cdots,y(t-\tau_m(t)))$ $(0<\varepsilon\ll 1)$的非线性多变延迟奇异摄动系统的理论解的稳定性,得到了系统稳定的一个充分条件.在此条件下还证明了隐式 Euler 方法的数值解是稳定的.

关键词　多变时滞奇异摄动问题; Euler 方法; 稳定性; 插值

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