Two conditions for a bipartite graph to be a k-deleted graph

Yang Hongchen Xue Xiuqian

(College of Sciences, China University of Mining and Technology, Xuzhou 221008, China)

Abstract: A k-regular spanning subgraph of graph G is called a k-factor of G. Graph G is called a k-deleted graph if $G-e$ has a k-factor for each edge e. A graph $G=(X,Y)$ with bipartition (X,Y) is called a bipartite graph if every edge of G has one endpoint in X and the other in Y. It is proved that a bipartite graph $G=(X,Y)$ with $|X|=|Y|$ is a k-deleted graph if and only if $k|S| \leqslant r_1+2r_2+\cdots+k(r_k+\cdots+r_\Delta)-\varepsilon(S)$ for all $S\subseteq X$. Using this result we give a sufficient neighborhood condition for a bipartite to be a k-deleted graph.

Key words: bipartite graph; k-factor; k-deleted graph

All graphs considered are simple, without loops or multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We denote by $d_G(v)$ the degree of a vertex v of G. The neighborhood $N_G(v)$ of vertex v is the set of vertices of G adjacent to v. For $S\subset V(G)$, $N_G(S)$ denotes the union of $N_G(v)$ over all $v\in S$. Let S and T be disjoint subsets of $V(G)$. Then $e_G(S,T)$ denotes the number of edges joining S and T in G. Particularly, $e_G(\{x\},S)$ denotes the number of edges joining a vertex x and S in G. A k-regular spanning subgraph of G is calleda k-factor of G. Graph G is called a k-deleted graph if $G-e$ has a k-factor for each edge e of G. All the other definitions and notation not mentioned in this article can be found in Ref.[1].

Liu Guizhen[2] studied the problem about (f,g)-deleted graph. Wang Changping[3] obtained several sufficient conditions for a graph to be a k-deleted graph. Qian Jianbo[4] discussed the relation between the existence of a k-factor and the degree condition in a bipartite graph. Chen Ciping[5] gave a neighborhood condition for a bipartite graph to have a k-factor. In this paper, we give a necessary and sufficient condition and a sufficient neighborhood condition for a bipartite graph to be a k-deleted graph. The following lemma will be used in our proof of theorems.

Lemma 1[6] Let $G=(X,Y)$ be a bipartite graph with $|X|=|Y|$ and $k\in\mathbf{N}$. Then G has a k-factor if and only if

$$k|S|\leqslant r_1+2r_2+\cdots+k(r_k+\cdots+r_\Delta)\qquad \text{for all } S\subseteq X$$

where $r_i=|R_i|$, $R_i=\{y\in Y|e_G(S,\{y\})=i\}, i=1,2,\cdots,\Delta$, and Δ is the maximum degree of G.

The following two theorems are our main results.

Theorem 1 Let $G=(X,Y)$ be a bipartite graph with $|X|=|Y|$ and $k\in\mathbf{N}$, then G is a k-deleted graph if and only if

$$k|S|\leqslant r_1+2r_2+\cdots+k(r_k+\cdots+r_\Delta)-\varepsilon(S)\qquad \text{for all } S\subseteq X \tag{1}$$

where $r_i=|R_i|$, $R_i=\{y\in Y|e_G(S,\{y\})=i\}$, $i=1,2,\cdots,\Delta$, Δ is the maximum degree of G, and

$$\varepsilon(S)=\begin{cases}1 & r_1+r_2+\cdots+r_k>0\\ 0 & \text{otherwise}\end{cases}$$

Proof Let $e=uv$ be any edge of G with $u\in X, v\in Y$, and set $H=G-e$. We now prove that graph H has a k-factor by using lemma 1.

Let S be any subset of X, and set $R'_i=\{y\in Y|e_H(S,\{y\})=i\}$, $r'_i=|R'_i|$, $i=1,2,\cdots,\Delta$.

We now consider two cases.

Case 1 $u\notin S$. Here $r'_i=r_i, i=1,2,\cdots,\Delta$ is obvious. By condition (1), we obtain

$$k|S|\leqslant r_1+2r_2+\cdots+k(r_k+\cdots+r_\Delta)-\varepsilon(S)\leqslant r_1+2r_2+\cdots+k(r_k+\cdots+r_\Delta)= r'_1+2r'_2+\cdots+k(r'_k+\cdots+r'_\Delta)$$

Case 2 $u\in S$. Let $e_G(\{v\},S)=t$ and $v\in R_t, 1\leqslant t\leqslant\Delta$.

If $k+1\leqslant t\leqslant\Delta$, then we have $r_k+\cdots+r_\Delta=r'_k+\cdots+r'_\Delta$ and $r_i=r'_i(i=1,2,\cdots,k-1)$.

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Biography: Yang Hongchen (1964—), male, lecturer, yanghongc@sina.com.

By condition (1), we obtain

$$k|S| \leqslant r_1 + 2r_2 + \cdots + k(r_k + \cdots + r_\Delta) - \varepsilon(S) \leqslant r_1 + 2r_2 + \cdots + k(r_k + \cdots + r_\Delta) = r'_1 + 2r'_2 + \cdots + k(r'_k + \cdots + r'_\Delta)$$

If $1 \leqslant t \leqslant k$, then we have $r_1 + r_2 + \cdots + r_k > 0$. This implies that $\varepsilon(S) = 1$, and so

$$r_1 + 2r_2 + \cdots + kr_k = r'_1 + 2r'_2 + \cdots + kr'_k + 1,\ r_i = r'_i \qquad i = k+1, \cdots, \Delta$$

By condition (1) we obtain

$$k|S| \leqslant r_1 + 2r_2 + \cdots + k(r_k + \cdots + r_\Delta) - 1 = r'_1 + 2r'_2 + \cdots + k(r'_k + \cdots + r'_\Delta) + 1 - 1 = r'_1 + 2r'_2 + \cdots + k(r'_k + \cdots + r'_\Delta)$$

The above proofs illustrate that

$$k|S| \leqslant r'_1 + 2r'_2 + \cdots + k(r'_k + \cdots + r'_\Delta) \quad \text{for all } S \subseteq X$$

Hence, graph H has a k-factor by lemma 1. This implies that graph G is a k-deleted graph.

Conversely, suppose (1) does not hold. Then there must exist a subset S of X such that

$$k|S| > r_1 + 2r_2 + \cdots + k(r_k + \cdots + r_\Delta) - \varepsilon(S) \tag{2}$$

We now distinguish two cases.

Case 1 $r_1 = r_2 = \cdots = r_k = 0$. If $r_1 = r_2 = \cdots = r_k = 0$, then $\varepsilon(S) = 0$, and so there exists a vertex $v \in R_{k+1} \cup R_{k+2} \cup \cdots \cup R_\Delta$. Let e be an edge joining v and S in G. Set $H = G - e$, and

$$R'_i = \{y \in Y \mid e_H(S, \{y\}) = i\},\ r'_i = |R'_i|,\ i = 1, 2, \cdots, \Delta$$

Clearly, $r'_k + \cdots + r'_\Delta = r_k + \cdots + r_\Delta$, and $r'_1 = r'_2 = \cdots = r'_{k-1} = 0$.

With (2) we get $k|S| > r_1 + 2r_2 + \cdots + k(r_k + \cdots + r_\Delta) = r'_1 + 2r'_2 + \cdots + k(r'_k + \cdots + r'_\Delta)$.

By lemma 1, graph H has no k-factor. This contradicts the theory that G is a k-deleted graph.

Case 2 $r_1 + r_2 + \cdots + r_k > 0$. If $r_1 + r_2 + \cdots + r_k > 0$, then $\varepsilon(S) = 1$, and so there exists a vertex $v \in R_1 \cup R_2 \cup \cdots \cup R_k$. Let e be an edge joining v and S in G. Set $H = G - e$, and

$$R'_i = \{y \in Y \mid e_H(S, \{y\}) = i\},\ r'_i = |R'_i|,\ i = 1, 2, \cdots, \Delta$$

Clearly, $r_1 + 2r_2 + \cdots + kr_k = r'_1 + 2r'_2 + \cdots + kr'_k + 1$, $r_i = r'_i\ (i = k+1, \cdots, \Delta)$.

With (2) we get

$$k|S| > r_1 + 2r_2 + \cdots + k(r_k + \cdots + r_\Delta) - 1 = r'_1 + 2r'_2 + \cdots + k(r'_k + \cdots + r'_\Delta) + 1 - 1 = r'_1 + 2r'_2 + \cdots + k(r'_k + \cdots + r'_\Delta)$$

By lemma 1, graph H has no k-factor. This contradicts that the theory G is a k-deleted graph. The proof of the theorem 1 is complete.

Theorem 2 Let $G = (X, Y)$ be a bipartite graph with $|X| = |Y| > 2k - 1$ and $k \in \mathbf{N}$, $k \neq 1$. If

$$|N_G(S)| \begin{cases} = |Y| & |S| \geqslant \dfrac{2}{2k-1}|X| - 2 \\ > \dfrac{2k-1}{2}|S| & \text{otherwise} \end{cases} \qquad \text{for all } S \subseteq X$$

Then G is a k-deleted graph.

Proof To the contrary, we assume G is not a k-deleted graph. By theorem 1 there must exist a minimal $S \subseteq X$ such that

$$k|S| > r_1 + 2r_2 + \cdots + k(r_k + \cdots + r_\Delta) - \varepsilon(S) \tag{3}$$

where $|r_i| = |R_i|$, $R_i = \{y \in Y \mid e_G(S, \{y\}) = i\}$, $i = 1, 2, \cdots, \Delta$, and Δ is the maximum degree of G, and

$$\varepsilon(S) = \begin{cases} 1 & r_1 + r_2 + \cdots + r_k > 0 \\ 0 & \text{otherwise} \end{cases}$$

We first show that for every $x \in S$

$$e_G(\{x\}, R_1 \cup R_2 \cup \cdots \cup R_k) \leqslant k - 1 \tag{4}$$

Suppose that there exists a vertex $x \in S$ such that $e_G(\{x\}, R_1 \cup R_2 \cup \cdots \cup R_k) \geqslant k$.

Set $S' = S - \{x\}$, and $R'_i = \{y \in Y \mid e_G(\{y\}, S') = i\}$, $r'_i = |R'_i|$, $i = 1, 2, \cdots, \Delta$. Then it is obvious that $r'_1 + 2r'_2 + \cdots + k(r'_k + \cdots + r'_\Delta) \leqslant r_1 + 2r_2 + \cdots + k(r_k + \cdots + r_\Delta) - k$.

With (3) we get $k|S'| = k(|S| - 1) = k|S| - k \geqslant r'_1 + 2r'_2 + \cdots + k(r'_k + \cdots + r'_\Delta) - \varepsilon(S')$.

This contradicts the minimum of S. Hence, (4) holds. So we obtain from (4) that

$$(k-1)|S| \geqslant e_G(S, R_1 \cup R_2 \cup \cdots \cup R_k) = r_1 + 2r_2 + \cdots + kr_k \tag{5}$$

Set $t = \min\{j \mid r_j \neq 0,\ j = 1,2,\cdots,\Delta\}$. We now show that $t \leqslant k$. Suppose $t \geqslant k+1$. Using (5) we obtain $k|S| > k(r_k + \cdots + r_\Delta)$, that is $|S| > r_k + \cdots + r_\Delta = N_G(S)$, which contradicts the condition of theorem 2. Hence, we have $t \leqslant k$.

Combining (3) and (5) we get (Note that $\varepsilon(S) = 1$)

$$(2k-1)|S| \geqslant 2r_1 + 4r_2 + \cdots + 2kr_k + k(r_{k+1} + \cdots + r_\Delta) \geqslant \min\{2t, k\}(r_1 + r_2 + \cdots + r_\Delta) = \min\{2t, k\} \cdot |N_G(S)|$$

therefore, $|N_G(S)| \leqslant \dfrac{2k-1}{2}|S|$.

Note the conditions of theorem 2. So, we get from the above estimate that $|N_G(S)| = |Y| = |X|$, and thus, $|S| \geqslant \dfrac{\min\{2t, k\}}{2k-1}|X|$. Let $v \in R_t$, and $S_0 = S - N_G(v)$. Then

$$|S_0| = |S| - t \geqslant \frac{\min\{2t, k\}}{2k-1}|X| - t \tag{6}$$

Case 1 $2t > k$. Since $t \leqslant k$, and $\dfrac{1}{2k-1}|X| - 1 > 0$, we have

$$|S_0| \geqslant \frac{k}{2k-1}|X| - t \geqslant \frac{k}{2k-1}|X| - k \geqslant \frac{2}{2k-1}|X| - 2$$

Case 2 $2t \leqslant k$. With (6) we get

$$|S_0| \geqslant \frac{2t}{2k-1}|X| - t > \frac{2}{2k-1}|X| - 2$$

But we have $|N_G(S_0)| \leqslant |Y - \{v\}| = |Y| - 1$, which contradicts the conditions of theorem 2. This contradiction completes the proof of theorem 2.

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二分图为 k-消去图的 2 个条件

杨宏晨　　薛秀谦

（中国矿业大学理学院，徐州 221008）

摘　要　图 G 的一个 k-正则支撑子图称为 G 的 k-因子. 若对 G 的任一边 e, 图 G 总存在一个 k-因子不含 e, 则称 G 是 k-消去图. 若图 G 存在一个划分 (X, Y) 使得 G 的每条边的端点分别在 X 和 Y 中, 则称 $G = (X, Y)$ 为二分图. 证明了二分图 $G = (X, Y)$ 且 $|X| = |Y|$ 是 k-消去图的充分必要条件是 $k|S| \leqslant r_1 + 2r_2 + \cdots + k(r_k + \cdots + r_\Delta) - \varepsilon(S)$ 对所有 $S \subseteq X$ 成立. 并由此给出二分图是 k-消去图的一个邻集充分条件.

关键词　二分图; k-因子; k-消去图

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