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On meromorphic functions sharing one value

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Abstract: The uniqueness of meromorphic functions with one sharing value and an equality on deficiency is studied. We show that if two nonconstant meromorphic functions $f(z)$ and $g(z)$ satisfy $\delta(0,f)+\delta(0,g)+\delta(\infty,f)+\delta(\infty,g)=3$ or $\delta_2(0,f)+\delta_2(0,g)+\delta_2(\infty,f)+\delta_2(\infty,g)=3$, and $E(1,f)=E(1,g)$ then $f(z),g(z)$ must be one of five cases.

Key words: meromorphic function; sharing value; deficiency

1 Introduction and the Main Result

In this paper, by meromorphic function we always mean a function which is meromorphic in the whole complex plane C. Let $f(z)$ be a meromorphic function, we shall use the following standard notations in Nevanlinna theory[1,2]:

$$T(r,f),m(r,f),N(r,f),\overline{N}(r,f)\cdots$$

We denote by $S(r,f)$ any function satisfying

$$S(r,f)=o\{T(r,f)\}$$

as $r\to+\infty$, possibly outside a set of finite Lebesgue measure.

Let a be a complex number. Set

$$E(a,f)=\{z\,|\,f(z)-a=0\}$$

where a zero point with multiplicity m is counted m times in the set.

Let $N_{(2}\left(r,\dfrac{1}{f-a}\right)$ be the counting function which only includes multiple zero of $f(z)-a$. Set

$$N_2\left(r,\frac{1}{f-a}\right)=\overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}_{(2}\left(r,\frac{1}{f-a}\right),\quad \delta_2(a,f)=1-\varlimsup_{\substack{r\to\infty\\ r\notin E}}\frac{N_2\left(r,\dfrac{1}{f-a}\right)}{T(r,f)}$$

$$\delta(a,f)=1-\varlimsup_{\substack{r\to\infty\\ r\notin E}}\frac{N\left(r,\dfrac{1}{f-a}\right)}{T(r,f)},\quad Ⓗ(r,f)=1-\varlimsup_{\substack{r\to\infty\\ r\notin E}}\frac{\overline{N}\left(r,\dfrac{1}{f-a}\right)}{T(r,f)}$$

where E is a set of r with finite measure. In this paper, E may be different at different places[3,4].

Theorem 1[5] Let $f(z)$ and $g(z)$ be nonconstant meromorphic functions satisfying

$$\delta(\infty,f)=\delta(\infty,g)=1,\ \delta(0,f)+\delta(0,g)>1$$

If $E(1,f)=E(1,g)$, then either $fg\equiv1$ or $f\equiv g$, and the number 1 in the above inequality is sharp.

The following result gives an improvement of theorem 1[6].

Theorem 2 Let $f(z)$ and $g(z)$ be nonconstant meromorphic functions satisfying

$$\delta(0,f)+\delta(0,g)+Ⓗ(\infty,f)+Ⓗ(\infty,g)>3$$

If $E(1,f)=E(1,g)$, $E(\infty,f)=E(\infty,g)$, then either $fg\equiv1$ or $f\equiv g$, and the number 3 in the above inequality is sharp.

Now, what conclusion can be made, if

$$\delta(0,f)+\delta(0,g)+\delta(\infty,f)+\delta(\infty,g)=3?$$

Theorem 3 Let $f(z)$ and $g(z)$ be nonconstant meromorphic functions satisfying

$$\delta(0,f)+\delta(0,g)+\delta(\infty,f)+\delta(\infty,g)=3 \text{ or } \delta_2(0,f)+\delta_2(0,g)+\delta_2(\infty,f)+\delta_2(\infty,g)=3 \quad (1)$$

If $E(1,f)=E(1,g)$, then one of the following cases must occur:

(a) $\delta(0,f)=\delta(\infty,f)=1$, $\delta(0,g)+\delta(\infty,g)=1$, $\delta(a,g)=0$ for any complex number $a(\neq1,0,\infty)$;

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(b) $\delta(0,f)=\delta(\infty,f)=1$, $\delta(0,g)+\delta(\infty,g)=1$, $\delta(a,f)=0$ for any complex number $a(\neq 1,0,\infty)$;

(c) $\delta(0,f)+\delta(\infty,f)>1$, $\delta(0,g)+\delta(\infty,g)>1$, and $\delta(a,f)=\delta(a,g)=0$, for any complex number $a(\neq 0,\infty)$;

(d) $f\equiv(1-a)g+a$, $a\neq 1$, in particular, if $a=0$, $f\equiv g$;

(e) $(f-a)(g-b)\equiv(1-a)(1-b)$, $a\neq 1$, $b\neq 1$.

Remark The following examples show that each of the above cases definitely occurs.

Example 1 Let $f(z)=e^z$, $g(z)=e^z(e^z-1)+1$, Obviously, $E(1,f)=E(1,g)$, $\delta(0,f)=\delta(\infty,f)=1$, $\delta(0,g)+\delta(\infty,g)=1$, $\delta(1,g)=\dfrac{1}{2}\neq 0$, $\delta(a,g)=0$, for any $a(\neq 0,1,\infty)$. Hence cases (a) and (b) occur.

Example 2 Let $f(z)=e^z(1-e^z)$, $g(z)=e^{-z}(1-e^{-z})$. Obviously, $E(1,f)=E(1,g)$, $\delta(0,f)+\delta(\infty,f)=\dfrac{3}{2}>1$, $\delta(0,g)+\delta(\infty,g)=\dfrac{3}{2}>1$, $\delta(a,f)=\delta(a,g)=0$, for any $a(\neq 0,\infty)$. Hence case (c) occurs.

Example 3 Let $f(z)=e^z$, $g(z)=\dfrac{1}{2}e^z+\dfrac{1}{2}$. Obviously, case (d) occurs.

Example 4 Let $f(z)=\dfrac{1}{2}ze^{-z}$, $g(z)=\dfrac{1}{2}+\dfrac{e^z}{z}$. Obviously, $f\left(g-\dfrac{1}{2}\right)=\dfrac{1}{2}$. Hence case (e) occurs.

2 One Lemma

To prove theorem 3, we need the following lemma.

Lemma Let $f(z)$ and $g(z)$ be nonconstant meromorphic functions satisfying

① $A=\sum_{a\neq 1}\delta_2(a,f)+\sum_{a\neq 1}\delta_2(a,g)+\min\{\delta_2(1,f),\delta_2(1,g)\}>3$

② $\sum_{a\neq 1}Ⓗ(a,f)>1$, $\sum_{a\neq 1}Ⓗ(a,g)>1$

If $E(1,f)=E(1,g)$, then one of the following cases must occur:

(a) $f\equiv(1-b)g+b$, $b\neq 1$;

(b) $(f-a)(g-b)\equiv(1-a)(1-b)$, where $a,b\in C$;

(c) $f\equiv\dfrac{Ag+B}{Cg+D}$, $AD-BC\neq 0$ and $Ⓗ(0,f)+Ⓗ(\infty,f)+Ⓗ(0,g)+Ⓗ(\infty,g)\leqslant 2$.

Proof By ② we deduce that

$$\sum_{i=1}^{q}Ⓗ(a_i,f)>1,\quad \sum_{i=1}^{q}Ⓗ(a_i,g)>1 \tag{2}$$

where $a_1,a_2,\cdots,a_q$ are pairwise distinct complex numbers such that $a_i\neq 1$ $(i=1,2,\cdots,q)$.

By the second fundamental theorem[7] we have

$$(q-1)T(r,f)\leqslant \overline{N}\left(r,\frac{1}{f-1}\right)+\sum_{i=1}^{q}\overline{N}\left(r,\frac{1}{f-a_i}\right)+S(r,f) \tag{3}$$

$$(q-1)T(r,g)\leqslant \overline{N}\left(r,\frac{1}{g-1}\right)+\sum_{i=1}^{q}\overline{N}\left(r,\frac{1}{g-a_i}\right)+S(r,g) \tag{4}$$

It can be obtained from (2)-(4) and $E(1,f)=E(1,g)$ that

$$T(r,f)=O\{T(r,g)\},\ T(r,g)=O\{T(r,f)\}\qquad r\to\infty;\ r\notin E \tag{5}$$

where E is a set of r with finite measure. Hence $S(r,f)=S(r,g)$.

Set

$$\varphi=\frac{f''}{f'}-2\frac{f'}{f-1}-\frac{g''}{g'}+2\frac{g'}{g-1} \tag{6}$$

Since $E(1,f)=E(1,g)$, by a simple computation we see that if z_0 is a simple zero of both $f(z)-1$ and $g(z)-1$, then $\varphi(z_0)=0$.

Next we shall prove that $\varphi(z)\equiv 0$. Suppose on the contrary that $\varphi(z)\not\equiv 0$, then

$$N_{1)}\left(r,\frac{1}{f-1}\right)=N_{1)}\left(r,\frac{1}{g-1}\right)\leqslant N\left(r,\frac{1}{\varphi}\right)\leqslant T(r,\varphi)+o(1)\leqslant N(r,\varphi)+S(r,f) \tag{7}$$

where $N_{1)}\left(r,\dfrac{1}{f-1}\right)$ is the counting function which only counts simple zeros of $f(z)-1$ in $\{z:|z|\leqslant r\}$.

Let $z_0 \in E(1,f)$, then by a simple computation we deduce from $E(1,f) = E(1,g)$ and (6) that $\varphi(z_0) \neq \infty$. Therefore poles of φ only occur at zeros of f' and g', and at multiple poles of f and g by (6).

Let $a_1, a_2, \cdots, a_n$ be pairwise distinct finite complex numbers such that $a_i \neq 1$ ($i = 1,2,\cdots, n$) and for some positive number $\varepsilon(< A - 3)$,

$$\sum_{i=1}^{n} \delta_2(a_i,f) + \delta_2(\infty,f) + \sum_{i=1}^{n} \delta_2(a_i,g) + \delta_2(\infty,g) + \min\{\delta_2(1,f),\delta_2(1,g)\} - 3 > \varepsilon \tag{8}$$

It can be obtained from(6) and (7) that

$$N_{1)}\left(r,\frac{1}{f-1}\right) \leqslant N_0\left(r,\frac{1}{f'}\right) + N_0\left(r,\frac{1}{g'}\right) + \overline{N}_{(2}(r,f) + \overline{N}_{(2}(r,g) + \sum_{i=1}^{n}\left[\overline{N}_{(2}\left(r,\frac{1}{f-a_i}\right) + \overline{N}_{(2}\left(r,\frac{1}{g-a_i}\right)\right] + S(r,f) \tag{9}$$

where $N_0\left(r,\frac{1}{f'}\right)$ only counts those zeros of f' but not those zeros of $(f-1)\prod_{i=1}^{n}(f-a_i)$.

By the second fundamental theorem, we have

$$nT(r,f) \leqslant \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f-1}\right) + \sum_{i=1}^{n}\overline{N}\left(r,\frac{1}{f-a_i}\right) - N_0\left(r,\frac{1}{f'}\right) + S(r,f) \tag{10}$$

and

$$nT(r,g) \leqslant \overline{N}(r,g) + \overline{N}\left(r,\frac{1}{g-1}\right) + \sum_{i=1}^{n}\overline{N}\left(r,\frac{1}{g-a_i}\right) - N_0\left(r,\frac{1}{g'}\right) + S(r,g) \tag{11}$$

It is easy to see that

$$\overline{N}\left(r,\frac{1}{f-1}\right) + \overline{N}\left(r,\frac{1}{g-1}\right) = 2\overline{N}\left(r,\frac{1}{f-1}\right) \leqslant N_{1)}\left(r,\frac{1}{f-1}\right) + N_2\left(r,\frac{1}{f-1}\right) \tag{12}$$

Combining (9) - (12), we obtain

$$n[T(r,f) + T(r,g)] \leqslant \sum_{i=1}^{n}\left[\overline{N}\left(r,\frac{1}{f-a_i}\right) + \overline{N}_{(2}\left(r,\frac{1}{f-a_i}\right)\right] + \overline{N}(r,f) + \sum_{i=1}^{n}\left[\overline{N}\left(r,\frac{1}{g-a_i}\right) + \overline{N}_{(2}\left(r,\frac{1}{f-a_i}\right)\right] + \overline{N}(r,g) + N_2\left(r,\frac{1}{f-1}\right) + \overline{N}_{(2}(r,f) + \overline{N}_{(2}(r,g) + S(r,f) \tag{13}$$

With no loss of generality, we assume that

$$T(r,g) \leqslant T(r,f) \qquad r \in I \tag{14}$$

where I is a set of infinite measure. By the definition of deficiency and $E(1,f) = E(1,g)$, we can deduce from (13) that as $r \to +\infty, r \notin E$,

$$\left[\sum_{i=1}^{n}\delta_2(a_i,f) + \delta_2(\infty,f) - 1 - \frac{\varepsilon}{4}\right]T(r,f) + \left[\sum_{i=1}^{n}\delta_2(a_i,g) + \delta_2(\infty,g) + \delta_2(1,g) - 2 - \frac{\varepsilon}{4}\right]T(r,g) \leqslant o\{T(r,f)\}$$

From (13) and the above inequality, we obtain that as $r \to \infty, r \in I \setminus E$,

$$\left[\sum_{i=1}^{n}\delta_2(a,f) + \delta_2(\infty,f) + \sum_{i=1}^{n}\delta_2(a_i,g) + \delta_2(\infty,g) + \min\{\delta_2(1,f),\delta_2(1,g)\} - 3 - \frac{\varepsilon}{2}\right]T(r,g) \leqslant o\{T(r,f)\} \tag{15}$$

Thus we deduce from (8), (15) and $S(r,f) = S(r,g)$ that $T(r,g) = o\{T(r,g)\}$, $r \to \infty$, $r \in I \setminus E$, a contradiction. Therefore, $\varphi(z) \equiv 0$. We deduce from (6) that

$$f \equiv \frac{Ag + B}{Cg + D} \tag{16}$$

where A, B, C and D are finite complex numbers satisfying $AD - BC \neq 0$.

Next we consider five cases:

Case 1 $C = 0$, then we get from (15) that $f \equiv ag + b, a \neq 0$.

If $f \neq 1$, then we obtain from $E(1,f) = E(1,g)$ that $g \neq 1$. Thus, by $\sum_{a \neq 1}\delta_2(a,f) \leqslant 1$ and $\sum_{a \neq 1}\delta_2(a,g) \leqslant 1$. We have

$$\sum_{a\neq 1}\delta_2(a,f)+\sum_{a\neq 1}\delta_2(a,g)+\min\{\delta_2(1,f),\delta_2(1,g)\}\leqslant 3$$

which contradicts ①. Therefore there exists z_0 such that $f(z_0)=g(z_0)=1$. Hence, we have $a+b=1$. Thus we obtain $f\equiv(1-b)g+b, b\neq 1$.

Case 2 $A=0$, then we get from (16) that $f(g-b)\equiv c, c\neq 0$.

From $E(1,f)=E(1,g)$, we have $c=1-b$.

Hence $f(g-b)\equiv 1-b, b\neq 1$.

Case 3 $B=0$ and $A\neq 0, C\neq 0, D\neq 0$, then we get from (16) that $f\equiv\dfrac{Ag}{Cg+D}$.

Thus $f-\dfrac{A}{C}=\dfrac{-AD/C}{Cg+D}$.

Hence we have $(f-a)(g-b)\equiv c, c\neq 0$. Then, we deduce from $E(1,f)=E(1,g)$ that $c=(1-a)(1-b)$. Thus, we obtain

$$(f-b)(g-b)\equiv(1-a)(1-b)\qquad a\neq 1; b\neq 1$$

Case 4 $D=0$ and $A\neq 0, B\neq 0, C\neq 0$, then we get from (15) that $f\equiv\dfrac{Ag+B}{Cg}$. Then, $f-\dfrac{A}{C}\equiv\dfrac{B}{Cg}$.

Thus, we obtain from $E(1,f)=E(1,g)$ and the above equality that $(f-a)g\equiv 1-a$.

Case 5 $A\neq 0, B\neq 0, C\neq 0$ and $D\neq 0$.

From (15), it is easy to see that

$$N(r,g)=N\left(r,\frac{1}{f-A/C}\right),\ N\left(r,\frac{1}{g}\right)=N\left(r,\frac{1}{f-B/D}\right)$$

By the second fundamental theorem we have

$$\begin{aligned}2T(r,f)\leqslant{}&\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-A/C}\right)+\overline{N}\left(r,\frac{1}{f-B/D}\right)=\\&\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}(r,g)+\overline{N}\left(r,\frac{1}{g}\right)+s(r,f)\leqslant\\&[4-Ⓗ(0,f)-Ⓗ(\infty,f)-Ⓗ(0,g)-Ⓗ(\infty,g)+\varepsilon]T(r,f)+S(r,f)\end{aligned}$$

where ε is a sufficiently small positive number.

If $Ⓗ(0,f)+Ⓗ(\infty,f)+Ⓗ(0,g)+Ⓗ(\infty,g)>2$, we can deduce from the above inequality that $T(r,f)\leqslant o\{T(r,f)\}, r\to\infty, r\in I$, a contradiction. Therefore

$$Ⓗ(0,f)+Ⓗ(\infty,f)+Ⓗ(0,g)+Ⓗ(\infty,g)\leqslant 2$$

This completes the proof of the lemma.

3 Proof of Theorem 3

In order to prove theorem 3, we consider three cases:

Case 1 $\delta(0,f)=\delta(\infty,f)=1$ and $\delta(0,g)+\delta(\infty,g)=1$.

If $\delta(a,g)=0$ for any complex member a $(\neq 1,0,\infty)$, then the conclusion (a) is valid.

If there exists a complex a $(\neq 0,\infty,1)$ such that $\delta(a,g)>0$ then, by lemma, we can easily obtain that the conclusion (d) or (e) is valid.

Case 2 $\delta(0,f)+\delta(\infty,f)=1$, $\delta(0,g)=\delta(\infty,g)=1$.

By the same reasoning with case 1 we can obtain that the conclusion (b), (c) or (e) is valid.

Case 3 $\delta(0,f)+\delta(\infty,f)>1$, and $\delta(0,g)+\delta(\infty,g)>1$.

If both $\delta(a,f)=0$ and $\delta(a,g)=0$, for any complex number $a(\neq 0,\infty)$ then the conclusion (c) is valid.

If there exists a complex member a $(\neq 1,0,\infty)$ such that max $\{\delta(a,f),\delta(a,g)\}>0$ or $\min\{\delta(1,f),\delta(1,g)\}>0$. By lemma, we can easily obtain that one of the conclusions (d) or (e) is valid.

If min $\{\delta(1,f),\delta(1,g)\}=0$, but $\max\{\delta(1,f),\delta(1,g)\}>0$, with no loss of generality, we can assume that $\delta(1,f)>0$, and $\delta(1,g)=0$.

If $\delta_2(1,g)>0$, by lemma, we can easily obtain that either the conclusion (d) or (e) is valid.

If $\delta_2(1,g)=0$, by using the same reasoning that was used to prove formula (13), we can also obtain $T(r,f)=O\{T(r,g)\}, T(r,g)=O\{T(r,f)\}, r\to\infty, r\notin E$, hence $S(r,f)=S(r,g)$. Set

$$\varphi(z)=\frac{f''}{f'}-2\frac{f'}{f-1}-\frac{g''}{g'}+2\frac{g'}{g-1}\tag{17}$$

Next we shall prove that $\varphi(z)\equiv 0$. Suppose on the contrary that $\varphi(z)\not\equiv 0$. Using the same reasoning that was used to prove formula (13) we have

$$T(r,f)+T(r,g)\leqslant N_2\left(r,\frac{1}{f}\right)+N_2(r,f)+N_2\left(r,\frac{1}{g}\right)+N_2(r,g)+N_2\left(r,\frac{1}{f-1}\right)+S(r,f) \quad (18)$$

Let $$\varlimsup_{\substack{r\to\infty\\ r\notin E}}\frac{T(r,f)}{T(r,g)}=\lambda \quad (19)$$

From $\delta_2(1,f)>0$ and $\delta_2(1,g)=0$, we get that $\lambda>1$.

From (18), (19) and $\delta_2(1,g)=0$, we have

$$\lambda+1=\varlimsup_{\substack{r\to\infty\\ r\notin E}}\frac{T(r,f)}{T(r,g)}+1\leqslant\varlimsup_{\substack{r\to\infty\\ r\notin E}}\frac{N_2\left(r,\frac{1}{f}\right)}{T(r,g)}+\varlimsup_{\substack{r\to\infty\\ r\notin E}}\frac{N_2(r,f)}{T(r,g)}\leqslant$$
$$[1-\delta_2(0,f)]\lambda+[1-\delta_2(\infty,f)]\lambda+[1-\delta_2(0,g)]+[1-\delta_2(\infty,g)]+1$$

Hence we obtain

$$[\delta_2(0,f)+\delta_2(\infty,f)-1]\lambda\leqslant 2-\delta_2(0,g)-\delta_2(\infty,g)$$

Since $\delta_2(0,f)+\delta_2(\infty,f)\geqslant\delta(0,f)+\delta(\infty,f)>1$ and $\lambda>1$, we obtain

$$\delta_2(0,f)+\delta_2(\infty,f)-1<2-\delta_2(0,g)-\delta_2(\infty,g)$$

That is $\delta_2(0,f)+\delta_2(\infty,f)+\delta_2(0,g)+\delta_2(\infty,g)<3$, which contradicts (1). Therefore we have $\varphi(z)\equiv 0$. Hence we deduce from (17) that

$$f\equiv\frac{Ag+B}{Cg+D}$$

where A,B,C and D are finite complex numbers satisfying $AD-BC\neq 0$.

If $f\neq 1$, by $\delta(0,f)+\delta(\infty,f)>1$, we have $\delta(0,f)+\delta(\infty,f)+\delta(1,f)>2$, which contradicts $\sum_a\delta(a,f)\leqslant 2$. Therefore, there exists z_0 such that $f(z_0)=g(z_0)=1$. According to the proof of lemma we can easily obtain that one of the conclusion (d) or (e) is valid.

The proof of theorem 3 is complete.

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分担一个值的亚纯函数

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摘　要　研究了分担一个值且具有一个亏量等式的亚纯函数的惟一性问题.讨论了对任何2个非常数亚纯函数 $f(z),g(z)$ 只要满足:$\delta(0,f)+\delta(0,g)+\delta(\infty,f)+\delta(\infty,g)=3$ 或者 $\delta_2(0,f)+\delta_2(0,g)+\delta_2(\infty,f)+\delta_2(\infty,g)=3$ 且 $E(1,f)=E(1,g)$，那么,$f(z),g(z)$必定具有5种情形之一.

关键词　亚纯函数; 分担值; 亏量

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