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# The optimal performance of a generalized Carnot cycle for a generalized heat transfer law

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**Abstract:** The finite time thermodynamic performance of a generalized Carnot cycle, in which the heat transfer between the working fluid and the heat reservoirs obeys the generalized law $Q \propto (\Delta T)^m$, is studied. The optimal configuration and the fundamental optimal relation between power and efficiency of the cycle are derived. Some special examples are discussed. The results can provide some theoretical guidance for the design a practical engine.

**Key words:** finite heat source; optimal configuration; generalized Carnot cycle

In the analysis of finite-time thermodynamics or entropy generation minimization[1-8], the basic thermodynamic model is the so-called "endoreversible Carnot engine", in which only the irreversibility of a finite rate of heat transfer is considered. A major objective of finite-time thermodynamics is to understand irreversible finite-time processes and to establish the general and natural bounds upon the efficiency or maximum work for such processes. Another primary goal of finite-time thermodynamics is to establish general operating principles (e.g., the path that the system should follow for maximum efficiency, work or chemical efficiency) for the system which serves as a model for real processes.

It is often the case in practice that the power is generated from heat, which is carried by a finite amount of materiel with finite heat capacity, rather than from the heat extracted from an isothermal and infinite reservoir. In the reversible (infinite-time) limit, the cycle, which extracts the maximum work from a finite heat source, is qualitatively different from the Carnot cycle, and its theoretical efficiency is considerably smaller[9]. For the endoreversible cycle, the research on the effect of the finite heat reservoir on the performance includes two aspects: The first is to determine the optimal performance of the given finite thermal capacity cycles, such as Carnot cycle[10,11], Rankine cycle[12], Brayton cycle[13-15], etc. The optimization of the first aspect may be carried out with the fixed heat input[10] or with the variable heat input[11-15]. The second is to determine the optimal configuration of heat engines with the given conditions. For example, the optimal configuration of an endoreversible constant-temperature heat reservoir heat engine is the Curzon-Ahlborn engine[16], and the optimal configuration of Newton's law system (heat transfer between working fluid and the heat reservoirs obeys $Q \propto (\Delta T)$) variable-temperature heat reservoir heat engine is a generalized endoreversible Carnot engine[17,18] in which the temperature of the heat reservoirs and the working fluids change exponentially with time and the ratio of the temperatures of the working fluid and the heat reservoir is a constant. The optimal performance of a generalized Carnot cycle for another linear heat transfer law $Q \propto (1/T)$ was studied in Ref.[19]. The effect of heat leak on the optimal configuration for Newton's law system heat engine is studied in Ref.[20]. These works found that the finite nature of the reservoirs is indeed an important feature which must be taken into account in many energy conversion systems of practical interest, because the optimal cycles and performances are different in systems with finite and nonisothemal reservoirs versus those with infinite isothermal reservoirs.

In this paper, the optimal configuration of a heat engine is studied further. It is assumed that the heat source has finite and constant heat capacity, that the only irreversibility arises from heat resistances, and that the heat transfer between working fluid and the heat reservoirs obeys the generalized transfer law $Q \propto (\Delta T)^{m}$ [21-25]. The fundamental optimal performance

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between the power output and efficiency of the generalized heat engine is derived in this paper. Some special examples are discussed. The results can provide some theoretical guidance for the design of practical engines.

## 1 Heat Engine Model

The generalized engine model and its surroundings to be considered in this paper are shown in Fig.1. The following assumptions are made for this model. The system adapted is a working fluid alternately connected to a hot source with finite heat capacity and to a cold sink with infinite heat capacity. The engine operates in a cyclic fashion with fixed time $\tau$ allotted for each cycle. The high-temperature heat source is assumed to have constant heat capacity $C$, its temperature is given by $T_x(t)$, and its initial temperature is given by $T_H$. For simplicity the cold sink is assumed to be infinite in size and therefore it has a fixed temperature $T_L$.

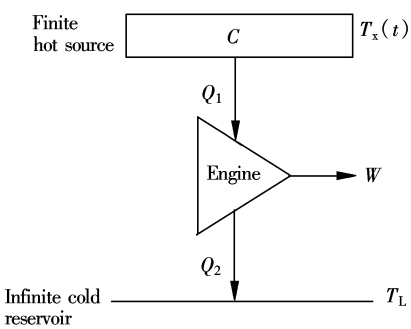


**Fig.1** The model of the engine

The heat transfer between the reservoirs and the working fluid obeys a generalized heat transfer law $Q \propto (\Delta T)^m$[21-25]. Therefore, one has

$$Q_1 = \int_0^\tau K_1(t)[T_x(t) - T(t)]^m \mathrm{d}t \tag{1}$$

$$Q_2 = \int_0^\tau K_2(t)[T(t) - T_L]^m \mathrm{d}t \tag{2}$$

where $T(t)$ is the temperature of the working fluid, $Q_i (i = 1,2)$ is the heat flux from the $i$-th reservoir to the working fluid and $K_i(t)(i = 1,2)$ is the thermal conductivity for heat transfer between the $i$-th reservoir and the working fluid. At $t = 0$, the working fluid is in contact with the high-temperature heat source and is separated from the low-temperature heat sink by an adiabatic boundary. At a later time $t_1 (0 < t_1 < \tau)$, contact with the heat source is broken, and the working fluid is placed in contact with the heat sink. Therefore, $K_1(t)$ and $K_1(t)$ can be written as

$$K_1(t) = \begin{cases} K_1 & 0 \leqslant t < t_1 \\ 0 & t_1 \leqslant t < \tau \end{cases} \tag{3}$$

$$K_2(t) = \begin{cases} 0 & 0 \leqslant t < t_1 \\ K_2 & t_1 \leqslant t < \tau \end{cases} \tag{4}$$

where $K_1$ and $K_2$ are constants.

From the first law of thermodynamics, the power produced is given by

$$\dot{W} = -\dot{E} + K_1(t)[T_x(t) - T(t)]^m - K_2(t)[T(t) - T_L]^m \tag{5}$$

where $E$ is the total energy of the working fluid. The total work produced in one cycle of duration $\tau$ is

$$W = \int_0^\tau \{K_1(t)[T_x(t) - T(t)]^m - K_2(t)[T(t) - T_L]^m\}\mathrm{d}t - \Delta E \tag{6}$$

where $\Delta E$ is the change in total energy of the working fluid. Since the engine works cyclically, $\Delta E$ over the full cycle must be zero. Hence, the total work done along one full cycle is

$$W = \int_0^{t_1} K_1(t)[T_x(t) - T(t)]^m \mathrm{d}t - \int_{t_1}^\tau K_2(t)[T(t) - T_L]^m \mathrm{d}t \tag{7}$$

The change rate of the entropy of the working fluid is

$$\dot{S} = \frac{K_1(t)[T_x(t) - T(t)]^m - K_2(t)[T(t) - T_L]^m}{T(t)} \tag{8}$$

Since the engine operates cyclically, the change in the entropy of the working fluid over one full cycle period is zero

$$\Delta S = \int_0^\tau \frac{K_1(t)[T_x(t) - T(t)]^m - K_2(t)[T(t) - T_L]^m}{T(t)}\mathrm{d}t = 0 \tag{9}$$

Furthermore, since the heat capacity of the hot source is assumed to be constant, one has

$$\mathrm{d}Q_1 = -C\mathrm{d}T_x(t) \tag{10}$$

Combining Eqs.(1) and (10) gives the constraint equation on the time rate of the change of the temperature of the high-temperature heat source in the following equation

$$C\dot{T}_x(t) + K_1(t)[T_x(t) - T(t)]^m = 0 \tag{11}$$

where $\dot{T}_x(t) = \mathrm{d}T_x(t)/\mathrm{d}t$.

## 2 Optimal Configuration

### 2.1 The generalized analytical solution

The problem now is to determine the optimal configuration of the model cycles in which the maximum work output is obtained under a given cycle time $\tau$. Using Eqs. (7), (9) and (11), one has the modified Lagrangian

$$L = K_1(t)[T_x(t) - T(t)]^m + K_2(t)[T_x(t) - T_L]^m +$$

$$\frac{\lambda\{K_1(t)[T_x(t)-T(t)]^m-K_2(t)[T(t)-T_L]^m\}}{T(t)}+$$
$$\mu\{C\dot{T}_x(t)+K_1(t)[T_x(t)-T(t)]^m\} \quad (12)$$

where $\lambda$ is a Lagrangian constant, and $\mu(t)$ is a function of time.

The path for the working fluid (as specified by $T(t)$ and $S(t)$), which results in the maximum work for a given time internal $\{0,\tau\}$, may now be obtained from the solution from the Euler-Lagrange equations.

The Euler-Lagrange equations are given by

$$\frac{\partial L}{\partial T(t)}-\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L}{\partial \dot{T}(t)}\right]=0 \quad (13)$$

and

$$\frac{\partial L}{\partial T_x(t)}-\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L}{\partial \dot{T}_x(t)}\right]=0 \quad (14)$$

Combining Eqs. (12), (13) and (14) gives:

$$[T_x(t)-T(t)]T(t)^{-\frac{2}{m+1}}=a(m) \qquad 0\leqslant t<t_1 \quad (15)$$

$$m-\lambda[T(t)-T_L]T^2(t)+\frac{m\lambda}{T(t)}=0 \qquad t_1\leqslant t<\tau \quad (16)$$

where $a(m)$ is a constant dependent on $m$.

Eqs. (15) and (16) are the major results of this paper. They determine the relation between the temperatures of heat reservoirs and the working fluid. Using Eqs. (11), (15) and (16), one can derive both the heat reservoirs and the working fluid temperatures versus time characteristic, i.e., the optimal configuration of the heat engine cycles.

### 2.2 Analysis for special cases

**Case 1** $m=1$

In this case, the heat transfer law obeys Newton's law. Combining Eqs. (11), (15) and (16) gives

$$T_x(t)=\begin{cases} T_H\exp\left[-(1-\mu)\dfrac{K_1}{C}t\right] & 0\leqslant t<t_1 \\ \nu T_L & t_1\leqslant t\leqslant\tau \end{cases} \quad (17)$$

and

$$T(t)=uT_x(t) \quad (18)$$

where $\nu$ and $u$ are constants. Eqs. (17) and (18) indicate that the temperatures of the high-temperature heat source and the working fluid decrease exponentially with time in the time interval $\{0,t_1\}$, and the ratio of the temperatures of the working fluid and heat source is a constant. This configuration is the same as the hat engine configuration obtained by Ondrechen, et al.[17] when $K_1=K_2$ and it can be called as a generalized endoreversible Carnot heat engine cycle.

**Case 2** $m=1.25$

Dulong and Petit found that the heat transfer coefficient is proportional to $(\Delta T)^{1/4}$ when they researched Newton's law of cooling in 1887, hence $Q\propto(\Delta T)^{1.25}$[22]. It is applied broadly in the heat transfer analysis in which forced convection is not dominant[22]. In this case, the exothermic process is still a constant temperature process. The varying laws of $T_x(t)$ and $T(t)$ in the heat absorbing process become complicate and follow the below relations

$$[T_x(t)-T(t)]T(t)^{-\frac{8}{9}}=a_2 \quad (19)$$

$$C\dot{T}_x(t)=K_1[T_x(t)-T(t)]^{\frac{5}{4}} \quad (20)$$

where $a_2$ is a constant.

## 3 Fundamental Optimal Relation

Combining the change in the entropy of the working fluid heat absorbing process

$$\mathrm{d}S_x=C\ln\left(1-\frac{Q_1}{CT_H}\right) \quad (21)$$

and the condition of internal reversible, one can introduce an equivalent temperature of hot reservoir

$$T_H^*=-\frac{Q_1}{\mathrm{d}S_x}=-\frac{Q_1}{C\ln\left(1-\dfrac{Q_1}{CT_H}\right)} \quad (22)$$

and an equivalent temperature of working fluid at heat-absorbing process

$$T_1^*=\frac{T_2Q_1}{Q_2} \quad (23)$$

where $T_2$ is the temperature of working fluid at exothermic process. Therefore, one can derive

$$Q_1=K_1(T_H^*-T_1^*)^m t_1 \quad (24)$$

$$Q_2=K_2(T_2-T_L)^m(\tau-t_1) \quad (25)$$

$$\eta=1-\frac{T_2}{T_1^*} \quad (26)$$

where $\eta$ is the efficiency of the heat engine cycle.

Combining Eqs. (22)-(26) gives the power output of the engine as

$$P=\frac{Q_1-Q_2}{\tau}=K_1\eta\left\{\frac{1}{T_H^*-T_1^*}+(1-\eta)\frac{K_1}{K_2}\frac{1}{[(1-\eta)T_1^*-T_L]^m}\right\}^{-1} \quad (27)$$

Taking the derivative of $P$ with respect to $T_1^*$ and setting it equal to zero yields

$$T_1^*=\frac{AT_H^*+T_L}{1-\eta+A} \quad (28)$$

Substituting Eq. (28) into Eq. (27) yields

$$P=K_1\eta\left(T_H^*-\frac{T_L}{1-\eta}\right)^m\cdot$$

$$\left[1 + (1 - \eta)^{\frac{1-m}{1+m}}\left(\frac{K_1}{K_2}\right)^{\frac{1}{1+m}}\right]^{-(1+m)} \tag{29}$$

where

$$A = \left[\frac{(1 + \eta)^2 K_1}{K_2}\right]^{\frac{1}{m+1}} \tag{30}$$

Eq.(29) is another major result of this paper. It determines the optimal relation between power output and efficiency for the fixed heat input $Q_1$. It is termed as the fundamental optimal relation for the generalized Carnot engine.

Taking the derivative of $P$ with respect to $\eta$ and setting it equal to zero yields

$$\eta_P = 1 - \frac{1}{2}\left\{\sqrt{\left[(m - 1)\frac{T_L}{T_H^*}\right]^2 + 4m\frac{T_L}{T_H^*}} - (m - 1)\frac{T_L}{T_H^*}\right\} \tag{31}$$

where $\eta_P$ is the efficiency bound of the heat engine at maximum power output $P_{max}$.

Since $T_H^*$ in Eqs.(29) and (30) is a function of $Q_1$, the fundamental optimal formula is related to the given $Q_1$. It is independent of $Q_1$ only if $C$ approaches infinite.

This result indicates that the influence of finite heat capacity rate heat source on the optimal performance of an endoreversible cycle can be expressed by an equivalent temperature, and that $T_H^*$ does not depend on the heat transfer law. The introduction of the equivalent temperature $T_H^*$ can turn the finite source cycle into an infinite source cycle when the optimal performance is discussed. In this case, however, the configuration of the cycle is not Carnot type because the heat-absorbing process is not an isothermal process. The temperature of the hot source varies with $f$ time. The optimal configuration is Carnot type only when $C \to \infty$ (i.e., $T_x(t) = T_H = a$ = constant).

The relations between optimal power and efficiency with different $m$ are shown in Fig.2 for fixed $Q_1$ with $K_1/K_2 = 1$, $T_H = 1\,200$ K, $T_L = 350$ K and $C = 10$ kJ/(kg·K). One can see that the bigger $m$ is, the smaller the efficiency is at $P = P_{max}$.

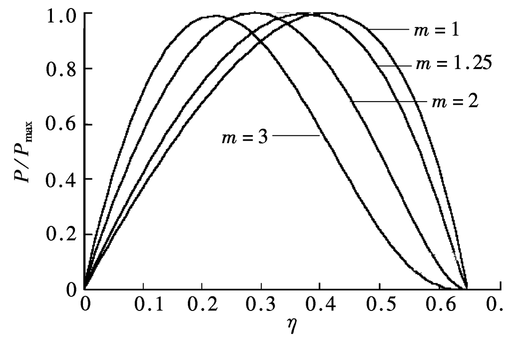


**Fig.2** The relation between $P/P_{max}$ and $\eta$

## 4 Conclusion

In practice, heat reservoirs are generally of finite size with finite heat capacity. Thus the problem of optimal configuration from which the maximum power is obtained with finite heat capacity is important. The finite time thermodynamic performance of a generalized Carnot cycle, in which the heat transfer between the working fluid and the heat reservoirs obeys a generalized law $Q \propto (\Delta T)^m$, is studied. The optimal configuration and the optimal relation between power and efficiency of the cycle are derived. The special examples provide comprehensive analysis of how heat transfer law influences the performance of the generalized heat engine. The results include those obtained in recent literatures and can provide some theoretical guidance for the design of practical engines.

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# 广义导热规律下广义卡诺循环的最优性能

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**摘　要**　研究了有限热源条件下工质与热源间传热规律服从 $Q\propto(\Delta T)^m$ 时广义卡诺热机的有限时间热力学性能，导出了循环的最优构形及最优功率与效率间的基本优化关系，给出了某些特例分析，所得结果对实际热机的设计工作具有一定的理论指导作用.

**关键词**　有限热源；最优构形；广义卡诺循环

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