Journal of Southeast University (English Edition) Vol.19 No.3 Sept. 2003 ISSN 1003—7985

Conserved quantities from Lie symmetries for nonholonomic systems

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Abstract: This paper presents a new method to seek the conserved quantity from a Lie symmetry without using either Lagrangians or Hamiltonians for nonholonomic systems. The differential equations of motion of the systems are established. The definition of the Lie symmetrical transformations of the systems is given, which only depends upon the infinitesimal transformations of groups for the generalized coordinates. The conserved quantity is directly constructed in terms of the Lie symmetry of the systems. The condition under which the Lie symmetry can lead to the conserved quantity and the form of the conserved quantity are obtained. Finally, an example is given to illustrate the application of the result.

Key words: analytical mechanics; nonholonomic system; symmetry; conserved quantity

A Lie symmetry is an invariance of the differential equations under infinitesimal transformations of groups. Under some conditions, the Lie symmetry can lead to a conserved quantity. In 1979, M. Lutzky introduced the Lie method to the mechanical field, and first studied the Lie symmetries and conserved quantities for dynamic systems[1]. Later, the method of Lie symmetry was developed rapidly, and especially in recent years, Prof. Mei has devoted himself to the study of the Lie symmetries for the constrained mechanical systems and has obtained a series of important results[2-7]. Recently, S. A. Hojman[8] and M. Lutzky[9,10] presented a new method for obtaining constants of motion from a non-Noether symmetry of a Lagrangian system. Y. Zhang[11,12] extended the method to the Birkhoffian systems and Hamiltonian systems.

In this paper, we further study the Lie symmetries and conserved quantities of the nonholonomic systems. Firstly, we give the Lie symmetries for which only the infinitesimal transformations of groups for the generalized coordinates are considered. Secondly, we present a method for obtaining the conserved quantities from the Lie symmetries of the systems; the method is independent of the Lagrangian of the systems and the structural equation of Lie symmetries (or Noether equality). Finally, an example is given to illustrate the application of the result.

Received 2002-11-22.
Foundation items: The National Natural Science Foundation of China (19972010) and the "Qinglan" Project Foundation of Jiangsu Province, China.
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1 Differential Equations of Motion of the System

Suppose that the configuration of a mechanical system is determined by n generalized coordinates q_s ($s = 1,\cdots,n$). The system is subjected to g ideal nonholonomic constraints of Chetaev's type.

$$f_\beta(t,\boldsymbol{q},\dot{\boldsymbol{q}}) = 0 \qquad \beta = 1,\cdots,g \tag{1}$$

The restriction of constraints (1) on the virtual displacements is[2]

$$\frac{\partial f_\beta}{\partial \dot{q}_s}\delta q_s = 0 \qquad \beta = 1,\cdots,g \tag{2}$$

From the D'Alembert-Lagrange's principle and Eq.(2), using the method of Lagrange's multiplier, we can obtain the equations of motion of the system in the Routh's form[2]

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q''_s + \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s} \qquad s = 1,\cdots,n \tag{3}$$

where L is the Lagrangian of the system; Q''_s are nonpotential generalized forces; λ_β are constraint multipliers. Assume that the system is non-singular, that is

$$\det(h_{sk}) = \det\left(\frac{\partial^2 L}{\partial \dot{q}_s \partial \dot{q}_k}\right) \neq 0 \tag{4}$$

From Eqs. (1) and (3) we can seek λ_β as the function of t, $\boldsymbol{q}, \dot{\boldsymbol{q}}$ before integrating the differential equations of motion, and thus Eq.(3) can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q''_s + \Lambda_s \qquad s = 1,\cdots,n \tag{5}$$

where

$$\Lambda_s = \Lambda_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}) = \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s} \tag{6}$$

Eq.(5) is called the differential equation of motion of the holonomic system corresponding to the nonholonomic system (1) and (3). When the initial conditions satisfy the nonholonomic constraints (1), the solution of Eq.(5) gives the motion of the nonholonomic system. Expanding Eq.(5), we can seek all the generalized accelerations as follows:

$$\ddot{q}_s = h_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}) \qquad s = 1,\cdots,n \tag{7}$$

2 Lie Symmetries of the System

In this paper, we consider the Lie symmetries of the system generated only by the infinitesimal transformations of generalized coordinates, that is to say, we can take the infinitesimal transformations of time and space as follows:

$$\left.\begin{aligned} &t^* = t \\ &q_s^*(t^*) = q_s(t) + \Delta q_s \qquad s = 1,\cdots,n \end{aligned}\right\} \tag{8}$$

or their expanded form

$$\left.\begin{aligned} &t^* = t \\ &q_s^*(t^*) = q_s(t) + \varepsilon\xi_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}) \qquad s = 1,\cdots,n \end{aligned}\right\} \tag{9}$$

Taking the infinitesimal generator vector $\boldsymbol{X}^{(0)}$ and its first extended vector $\boldsymbol{X}^{(1)}$ as

$$\boldsymbol{X}^{(0)} = \xi_s \frac{\partial}{\partial q_s},\ \boldsymbol{X}^{(1)} = \boldsymbol{X}^{(0)} + \dot{\xi}_s \frac{\partial}{\partial \dot{q}_s} \tag{10}$$

According to the Lie theory for the invariance of differential equations under infinitesimal transformations, the invariance of Eq.(7) under infinitesimal transformations (9) leads to the satisfaction of the following equation

$$\ddot{\xi}_s - \boldsymbol{X}^{(1)}(h_s) = 0 \qquad s = 1,\cdots,n \tag{11}$$

Eq.(11) can be called the determining equation of the system. Thus we have the following propositions 1 to 3.

Proposition 1 If the infinitesimal generators ξ_s satisfy the determining Eq.(11), then the corresponding symmetries are the Lie symmetries of a holonomic system (7) corresponding to the nonholonomic system (1) and (3).

The invariance of nonholonomic constraints Eq.(1) under infinitesimal transformations (9) leads to the satisfaction of the following restriction equations

$$\boldsymbol{X}^{(1)}(f_\beta(t,\boldsymbol{q},\dot{\boldsymbol{q}})) = 0 \qquad \beta = 1,\cdots,g \tag{12}$$

Proposition 2 If the infinitesimal generators ξ_s satisfy the determining Eq.(11) and the restriction Eq. (12), then the corresponding symmetries are the weak Lie symmetries of the nonholonomic system (1) and (3).

The Chetaev's conditions (2) of virtual displacements impose restrictions on the infinitesimal generators ξ_s, and we have

$$\frac{\partial f_\beta}{\partial \dot{q}_s}\xi_s = 0 \qquad \beta = 1,\cdots,g \tag{13}$$

Eq.(13) can be called the additional restriction equation.

Proposition 3 If the infinitesimal generators ξ_s satisfy the determining Eq.(11), the restriction Eq.(12) and the additional restriction Eq.(13), then the corresponding symmetries are the strong Lie symmetries of the nonholonomic system (1) and (3).

3 Conserved Quantities of the System

Lie symmetries do not always generate conserved quantities. The following propositions give the conditions under which the Lie symmetries can lead to a new type of conserved quantities, and the form of the conserved quantities.

Proposition 4 For the infinitesimal generators ξ_s satisfying the determining Eq.(11), if there exists a function $G = G(\boldsymbol{q},\dot{\boldsymbol{q}})$ satisfying the condition

$$\frac{\partial h_s}{\partial \dot{q}_s} + \frac{1}{G}\frac{\partial G}{\partial q_s}\dot{q}_s + \frac{1}{G}\frac{\partial G}{\partial \dot{q}_s}h_s = 0 \tag{14}$$

Then the holonomic system (7) corresponding to the nonholonomic system has a conserved quantity of Lie symmetry, such that

$$I = \frac{1}{G}\frac{\partial}{\partial q_s}(G\xi_s) + \frac{1}{G}\frac{\partial}{\partial \dot{q}_s}(G\dot{\xi}_s) = \text{const} \tag{15}$$

Proof

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}t} = &\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{G}\frac{\partial G}{\partial q_s}\right)\xi_s + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{G}\frac{\partial G}{\partial \dot{q}_s}\right)\dot{\xi}_s + \\ &\frac{1}{G}\frac{\partial G}{\partial q_s}\dot{\xi}_s + \frac{1}{G}\frac{\partial G}{\partial \dot{q}_s}\ddot{\xi}_s + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \xi_s}{\partial q_s} + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \dot{\xi}_s}{\partial \dot{q}_s}\end{aligned} \tag{16}$$

Using Eq.(10), Eq.(11) can be written as

$$\ddot{\xi}_s - \frac{\partial h_s}{\partial q_k}\xi_k - \frac{\partial h_s}{\partial \dot{q}_k}\dot{\xi}_k = 0 \qquad s,k = 1,\cdots,n \tag{17}$$

Hence, we have

$$\begin{aligned}&\frac{\partial \ddot{\xi}_s}{\partial \dot{q}_s} - \frac{\partial \xi_k}{\partial \dot{q}_s}\frac{\partial h_s}{\partial q_k} - \xi_k\frac{\partial^2 h_s}{\partial q_k \partial \dot{q}_s} - \\ &\quad \frac{\partial \dot{\xi}_k}{\partial \dot{q}_s}\frac{\partial h_s}{\partial \dot{q}_k} - \dot{\xi}_k\frac{\partial^2 h_s}{\partial \dot{q}_k \partial \dot{q}_s} = 0\end{aligned} \tag{18}$$

It is easy to verify

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \xi_s}{\partial q_s} = \frac{\partial \dot{\xi}_s}{\partial q_s} - \frac{\partial \xi_s}{\partial \dot{q}_k}\frac{\partial h_k}{\partial q_s} \tag{19}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \dot{\xi}_s}{\partial \dot{q}_s} = \frac{\partial \ddot{\xi}_s}{\partial \dot{q}_s} - \frac{\partial \dot{\xi}_s}{\partial q_s} - \frac{\partial \dot{\xi}_s}{\partial \dot{q}_k}\frac{\partial h_k}{\partial \dot{q}_s} \tag{20}$$

Substituting Eqs.(17)-(20) into Eq.(16), we obtain

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{G}\frac{\partial G}{\partial q_s}\right)\xi_s + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{G}\frac{\partial G}{\partial \dot{q}_s}\right)\dot{\xi}_s - \frac{\partial}{\partial q_k}\left(\frac{1}{G}\frac{\partial G}{\partial q_s}\right)\dot{q}_s\xi_k - \frac{\partial}{\partial q_k}\left(\frac{1}{G}\frac{\partial G}{\partial \dot{q}_s}\right)h_s\xi_k - \frac{\partial}{\partial \dot{q}_k}\left(\frac{1}{G}\frac{\partial G}{\partial q_s}\right)\dot{q}_s\dot{\xi}_k - \frac{\partial}{\partial \dot{q}_k}\left(\frac{1}{G}\frac{\partial G}{\partial \dot{q}_s}\right)h_s\dot{\xi}_k \tag{21}$$

Take notice of the relations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{G}\frac{\partial G}{\partial q_s}\right) = \frac{\partial}{\partial q_k}\left(\frac{1}{G}\frac{\partial G}{\partial q_s}\right)\dot{q}_k + \frac{\partial}{\partial \dot{q}_k}\left(\frac{1}{G}\frac{\partial G}{\partial q_s}\right)h_k \tag{22}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{G}\frac{\partial G}{\partial \dot{q}_s}\right) = \frac{\partial}{\partial q_k}\left(\frac{1}{G}\frac{\partial G}{\partial \dot{q}_s}\right)\dot{q}_k + \frac{\partial}{\partial \dot{q}_k}\left(\frac{1}{G}\frac{\partial G}{\partial \dot{q}_s}\right)h_k \tag{23}$$

and we can easily prove the following relations

$$\frac{\partial}{\partial q_k}\left(\frac{1}{G}\frac{\partial G}{\partial q_s}\right)\dot{q}_k\xi_s = \frac{\partial}{\partial q_k}\left(\frac{1}{G}\frac{\partial G}{\partial q_s}\right)\dot{q}_s\xi_k \tag{24}$$

$$\frac{\partial}{\partial \dot{q}_k}\left(\frac{1}{G}\frac{\partial G}{\partial q_s}\right)h_k\xi_s = \frac{\partial}{\partial q_k}\left(\frac{1}{G}\frac{\partial G}{\partial \dot{q}_s}\right)h_s\xi_k \tag{25}$$

$$\frac{\partial}{\partial q_k}\left(\frac{1}{G}\frac{\partial G}{\partial \dot{q}_s}\right)\dot{q}_k\dot{\xi}_s = \frac{\partial}{\partial \dot{q}_k}\left(\frac{1}{G}\frac{\partial G}{\partial q_s}\right)\dot{q}_s\dot{\xi}_k \tag{26}$$

$$\frac{\partial}{\partial \dot{q}_k}\left(\frac{1}{G}\frac{\partial G}{\partial \dot{q}_s}\right)h_k\dot{\xi}_s = \frac{\partial}{\partial \dot{q}_k}\left(\frac{1}{G}\frac{\partial G}{\partial \dot{q}_s}\right)h_s\dot{\xi}_k \tag{27}$$

Substituting Eqs. (22) -(27) into Eq. (21), we have

$$\frac{\mathrm{d}I}{\mathrm{d}t} = 0 \tag{28}$$

Therefore, the corresponding holonomic system (7) has the conserved quantity of Lie symmetry in the form of (15).

Proposition 5 For the infinitesimal generators ξ_s satisfying the determining Eq. (11) and the restriction Eq. (12), if there exists a function $G = G(\boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfying the condition (14), then the nonholonomic system has a conserved quantity of weak Lie symmetry in the form of (15).

Proposition 6 For the infinitesimal generators ξ_s satisfying the determining Eq. (11), the restriction Eq. (12) and the additional restriction Eq. (13), if there exists a function $G = G(\boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfying the condition (14), then the nonholonomic system has a conserved quantity of strong Lie symmetry in the form of (15).

It is necessary to point out that the three propositions above give a new method to seek conserved quantities from Lie symmetries of nonholonomic mechanical systems. The conserved quantity (15) depends on neither the Lagrangian of the system nor the structure equation (or Noether's equality) of usual Lie symmetries.

4 Example

Suppose the configuration of a system is determined by generalized coordinates q_1, q_2 and q_3, the Lagrangian of the system is

$$L = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) \tag{29}$$

The nonholonomic constraint is

$$f = \dot{q}_2 - \dot{q}_1\tan q_3 = 0 \tag{30}$$

and non-potential generalized forces are

$$\left.\begin{aligned} Q''_1 &= \frac{m\dot{q}_1\dot{q}_3\tan q_3}{\cos^2 q_3} \\ Q''_2 &= 0 \\ Q''_3 &= 0 \end{aligned}\right\} \tag{31}$$

Let us try to study its Lie symmetries and conserved quantities.

Firstly, we establish the differential equations of motion of the system. Eq. (3) give

$$\left.\begin{aligned} m\ddot{q}_1 &= -\lambda\tan q_3 + \frac{m\dot{q}_1\dot{q}_3\tan q_3}{\cos^2 q_3} \\ m\ddot{q}_2 &= \lambda \\ m\ddot{q}_3 &= 0 \end{aligned}\right\} \tag{32}$$

From Eq. (30) and Eq. (32), we have

$$\lambda = \frac{m\dot{q}_1\dot{q}_3}{\cos^2 q_3} \tag{33}$$

Hence, Eq. (32) come to the form

$$\left.\begin{aligned} \ddot{q}_1 &= 0 \\ \ddot{q}_2 &= \frac{\dot{q}_1\dot{q}_3}{\cos^2 q_3} \\ \ddot{q}_3 &= 0 \end{aligned}\right\} \tag{34}$$

Secondly, we study the Lie symmetries of the system. The determining Eq. (11) give

$$\left.\begin{aligned} \ddot{\xi}_1 &= 0 \\ \ddot{\xi}_2 &= \dot{\xi}_1\frac{\dot{q}_3}{\cos^2 q_3} + \dot{\xi}_3\frac{\dot{q}_1}{\cos^2 q_3} + \xi_3\frac{2\dot{q}_1\dot{q}_3\tan q_3}{\cos^2 q_3} \\ \ddot{\xi}_3 &= 0 \end{aligned}\right\} \tag{35}$$

Eq. (35) has the following solutions

$$\left.\begin{aligned} \xi_1 &= 1 \\ \xi_2 &= 0 \\ \xi_3 &= 0 \end{aligned}\right\} \tag{36}$$

$$\left.\begin{aligned} \xi_1 &= 0 \\ \xi_2 &= \dot{q}_2 \\ \xi_3 &= \dot{q}_3 \end{aligned}\right\} \tag{37}$$

The generators (36) and (37) correspond to the Lie symmetries of the holonomic system corresponding to the nonholonomic system.

The restriction Eq. (12) gives

$$\dot{\xi}_2 - \dot{\xi}_1 \tan q_3 - \xi_3 \frac{\dot{q}_1}{\cos^2 q_3} = 0 \tag{38}$$

Obviously, the generators (36) and (37) satisfy Eq.(38), thus they correspond to weak Lie symmetries of the nonholonomic system. The additional restriction Eq.(13) gives

$$\xi_2 - \xi_1 \tan q_3 = 0 \tag{39}$$

The generators (36) and (37) do not satisfy Eq.(39), therefore, the symmetries are not strong Lie symmetries of the nonholonomic system.

Finally, we seek the conserved quantities. From Eq.(14), we have

$$\frac{1}{G}\frac{\partial G}{\partial q_1}\dot{q}_1 + \frac{1}{G}\frac{\partial G}{\partial q_2}\dot{q}_2 + \frac{1}{G}\frac{\partial G}{\partial q_3}\dot{q}_3 + \frac{1}{G}\frac{\partial G}{\partial \dot{q}_2}\frac{\dot{q}_1\dot{q}_3}{\cos^2 q_3} = 0 \tag{40}$$

Eq.(40) has the following solutions

$$\ln G_1 = \frac{\dot{q}_1}{\dot{q}_3}(q_1\dot{q}_3 - \dot{q}_1 q_3) \tag{41}$$

$$\ln G_2 = \frac{\dot{q}_3}{\dot{q}_1}(\dot{q}_1 q_3 - q_1\dot{q}_3) \tag{42}$$

Substituting the generator (36) and the function G_1 into Eq.(15), we obtain

$$I_1 = \dot{q}_1 = \text{const} \tag{43}$$

Substituting the generator (37) and the function G_2 into Eq.(15) we get

$$I_2 = \dot{q}_3^2 = \text{const} \tag{44}$$

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非完整系统的 Lie 对称性守恒量

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摘　要　提出了由非完整系统的 Lie 对称性求守恒量的一种新方法,该方法不依赖于系统的 Lagrangian 函数或 Hamiltonian 结构.建立了系统的运动微分方程,给出了系统仅依赖于广义坐标的无限小群变换的 Lie 对称变换的定义,并直接由系统的 Lie 对称性构造守恒量,得到了 Lie 对称性导致守恒量的条件及守恒量的形式.最后举例说明结果的应用.

关键词　分析力学;非完整系统;对称性;守恒量

中图分类号　O316