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The smash products of entwining structure

Guo Guangquan

(Department of Mathematics, Nanjing Xiaozhuang College, Nanjing 210017, China)

Abstract: Let k be a commutative ring, C a projective k-coalgebra. The smash products of entwining structure $(A, C)_\psi$ are discussed. When the map ψ is a bijective, and C is a finitely generated k-module, a version of the Ulbrich theorem for coalgebras C is given.

Key words: coalgebra; entwining structure; smash product

Entwining structure $(A, C)_\psi$ and $(A, C)_\psi$-module are generalizations of Doi-Koppinen datumand and Doi-Koppinen Hopf modules. They have been generally discussed in recent years[1-3]. The smash product of entwining structure was discussed by Brzezinski[4]. In this paper, we study further the smash product of entwining structure, and we give a version of the Ulbrich theorem for coalgebras C.

Throughout the following, C denotes a coalgebra over commutative ring k, and C is a projective k-module; $C^* = \mathrm{Hom}(C, k)$ denote the k-algebra with the usual convolution $*$. Unless otherwise stated, all maps are k-linear, $\otimes$ means $\otimes_k$, Hom means Hom_k. We follow the notation in Ref. [5]. We shall frequently adopt the notations $\sum a_{(0)} \otimes a_{(1)}$, $\psi(c \otimes a) = a_\alpha \otimes c^\alpha$, etc.

1 The Entwined Structure and Smash Product

Definition 1[2] An entwining structure (over k) is a triple (A, C, ψ) consisting of a k-algebra A, a k-coalgebra C and a k-module map $\psi: C \otimes A \to A \otimes C$, (we use the notation $\psi(c \otimes a = a_\alpha \otimes c^\alpha)$ satisfying ① $(ab)_\alpha \otimes c^\alpha = a_\alpha b_\beta \otimes c^{\alpha,\beta}$; ② $a_\alpha \otimes \varepsilon(c^\alpha) = \varepsilon(c)a$; ③ $a_\alpha \otimes (c^\alpha)_1 \otimes (c^\alpha)_2 = a_{\alpha,\beta} \otimes (c_1)^\beta \otimes (c_2)^\alpha$; ④ $1_\alpha \otimes c^\alpha = 1 \otimes c$.

Example 1[2] Let H be a bialgebra, C a right H-module coalgebra, and A right H-comodule algebra. Then C and A are entwining by $\psi: C \otimes A \to A \otimes C$, $\psi(c \otimes a) = a_{(0)} \otimes ca_{(1)}$. Note that if H has a bijective antipode S, then ψ is a bijective with $\psi^{-1}: a \otimes c = cS^{-1}a_{(1)} \otimes a_{(0)}$.

When ψ is a bijective, we use the notation $\psi^{-1}(a \otimes c) = {}_\alpha c \otimes {}^\alpha a$. Then ψ^{-1} is k-linear and satisfies

$$\left.\begin{aligned} &{}_\alpha c \otimes {}^\alpha(ab) = {}_{\beta,\alpha}c \otimes ({}^\beta a)({}^\alpha b),\ \varepsilon({}_\alpha c) \otimes {}^\alpha(a) = \varepsilon(c)a \\ &({}_\alpha c)_1 \otimes ({}_\alpha c)_2 \otimes {}^\alpha(a) = {}_\alpha(c_1) \otimes {}_\beta(c_2) \otimes {}^{\beta,\alpha}a,\ {}_\alpha c \otimes {}^\alpha 1 = c \otimes 1 \end{aligned}\right\} \tag{1}$$

$${}_\beta(c^\alpha) \otimes {}^\beta(a_\alpha) = c \otimes a,\ ({}^\alpha a)_\beta \otimes ({}_\alpha c)^\beta = a \otimes c \tag{2}$$

If ψ is a bijective, the multiplication on $\mathrm{Hom}(C, A)$ is given by the formula

$$(f \# g)(c) = f[{}_\alpha(c_{(2)})][{}^\alpha g(c_{(1)})]$$

for all k-linear $f, g \in \mathrm{Hom}(C, A)$ and $c \in C$.

Since

$$[(f \# g) \# h](c) = (f \# g)[{}_\alpha(c_{(2)})][{}^\alpha h(c_{(1)})] = f\{{}_\beta[({}_\alpha(c_{(2)}))_{(2)}]\}\{{}^\beta g[({}_\alpha(c_{(2)}))_{(1)}]\}[{}^\alpha h(c_{(1)})] =$$
$$f[{}_\beta({}_\gamma(c_{(2)(2)}))]\{{}^\beta g[{}_\alpha(c_{(2)(1)})]\}[{}^{\gamma\alpha} h(c_{(1)})] = f[{}_\beta\,{}_\gamma(c_{(3)})]\{{}^\beta g[{}_\alpha(c_{(2)})]\}[{}^{\gamma\alpha} h(c_{(1)})]$$

and

$$[f \# (g \# h)](c) = f[{}_\beta(c_{(2)})][{}^\beta(g \# h)(c_{(1)})] = f[{}_\beta(c_{(3)})]\{{}^\beta[g({}_\alpha(c_{(2)}))\,{}^\alpha h(c_{(1)})]\} =$$
$$f[{}_\beta\,{}_\gamma(c_{(3)})]\{{}^\beta g[{}_\alpha(c_{(2)})]\}[{}^{\gamma\alpha} h(c_{(1)})]$$

for k-linear and $f, g \in \mathrm{Hom}(C, A)$, $c \in C$, $(f \# g) \# h = f \# (g \# h)$. Hence $\mathrm{Hom}(C, A)$ is an algebra with the identity element $\eta \circ \varepsilon$. In this case, we denote the k-algebra by $\#(C, A, \psi^{-1})$.

Assume that C is a finitely generated and projective k-module, and consider the entwining structure $(A, C)_\psi$,

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Biography: Guo Guangquan (1954—), male, associate professor, guogq1954@yahoo.com.cn.

the multiplication on k-space $A \otimes C^*$ is given by the formula

$$(a \otimes c^*)(b \otimes d^*) = a({}^{\alpha}b) \otimes c^*({}_{\alpha}c_i)(d^* * c_i^*)$$

for all $a, b \in A, c^*, d^* \in C^*$. It is easy to check that $A \otimes C^*$ is a k-algebra with an identity element $1 \otimes \varepsilon$. The k-algebra is denoted by $A \#_{\psi^{-1}} C^*$ and the element in $A \#_{\psi^{-1}} C^*$ is denoted by $a \# c^*$, $a \in A$, $c^* \in C^*$.

Let H be a finitely generated projective k-Hopf algebra, A a right H-comodule algebra. It is well known that $\#(H, A) \cong A \# H^*$ as algebras. This result was generalized to the smash product of Doi-Koppinen datum[6]. In this section, we will prove that $\#(C, A, \psi^{-1}) \cong A \#_{\psi^{-1}} C^*$.

We claim easily that the maps

$$\pi: A \to \#(C, A, \psi^{-1}),\ \pi(a)(c) = a\varepsilon(c)$$

$$\lambda: C^{*\mathrm{op}} \to \#(C, A, \psi^{-1}), \lambda(c^*)(c) = c^*(c)1_A$$

are homomorphisms of k-algebras.

Lemma 1 Let $(A, C)_{\psi}$ be an entwining structure and ψ a bijective. Then ① $[\pi(a) \# \lambda(c^*)](c) = ac^*(c)$; ②$[\lambda(c^*) \# \pi(a)](c) = c^*({}_{\alpha}c)({}^{\alpha}a)$ for all $a \in A, c^* \in C^*, c \in C$.

Proof For all $a \in A, c^* \in C^*, c \in C$, we have

$$[\pi(a) \# \lambda(c^*)](c) = \pi(a)({}_{\alpha}(c_{(2)}))[{}^{\alpha}(\lambda(c^*)(c_{(1)}))] = a\varepsilon({}_{\alpha}(c_{(2)}))[{}^{\alpha}(\lambda(c^*)(c_{(1)}))] = a\varepsilon(c_{(2)})c^*(c_{(1)}) = ac^*(c)$$

$$[\lambda(c^*) \# \pi(a)](c) = \lambda(c^*)({}_{\alpha}(c_{(2)}))[{}^{\alpha}(\pi(a)(c_{(1)}))] = c^*({}_{\alpha}(c_{(2)}))[({}^{\alpha}a)\varepsilon(c_{(1)}))] = c^*({}_{\alpha}c)({}^{\alpha}a)$$

Lemma 2 Assume that C is a finitely generated projective k-module, and let $\{c_i, c_i^*\}_{i=1}^n$ be a finite dual basis. Then for all $a \in A$, we have

$$[\lambda(c_i^*) \# (\pi(a)] \otimes c_i = \pi({}^{\alpha}a) \# \lambda(c_i^*) \otimes {}_{\alpha}(c_i)$$

Proof For all $x \in C, a \in A$, since $x = \sum_{i=1}^{n} c_i c_i^*(x)$, ${}_{\alpha}x = \sum_{i=1}^{n} c_i c_i^*({}_{\alpha}x)$,

$$\psi^{-1}(a \otimes x) = \psi^{-1}(a \otimes c_i c_i^*(x)) = \sum_{i=1}^{n} c_i^*(x)({}_{\alpha}(c_i) \otimes {}^{\alpha}a) = \sum_{i=1}^{n}({}_{\alpha}(c_i))c_i^*(x) \otimes {}^{\alpha}a = {}_{\alpha}x \otimes {}^{\alpha}a$$

Therefore

$$[\lambda(c_i^*) \# \pi(a)](x) \otimes c_i = ({}^{\alpha}a) \otimes c_i^*({}_{\alpha}x)c_i = {}^{\alpha}a \otimes {}_{\alpha}x = ({}^{\alpha}a) \otimes {}_{\alpha}(c_i^*(x)c_i) = ({}^{\alpha}a)\ c_i^*(x) \otimes {}_{\alpha}(c_i) = [\pi({}^{\alpha}a) \# \lambda(c_i^*)](x) \otimes {}_{\alpha}(c_i)$$

By proposition 20.10 in Ref.[7], we have

$$\lambda(c_i^*) \# \pi(a) \otimes c_i = \sum \pi({}^{\alpha}a) \# \lambda(c_i^*) \otimes {}_{\alpha}(c_i)$$

Corollary 1 Assume that C is finitely generated and projective as a k-module, and let $\{c_i, c_i^*\}_{i=1}^n$ be a finite dual basis. Then for all $a \in A, d^* \in C^*$, we have

$$\lambda(d^*) \# \pi(a) = \pi({}^{\alpha}a) \# \lambda(c_i^*)d^*({}_{\alpha}(c_i))$$

Proof For all $a \in A, d^* \in C^*$, we have

$$(1 \otimes d^*)[\lambda(c_i^*) \# (\pi(a) \otimes c_i] = (1 \otimes d^*)[\pi({}^{\alpha}a) \# \lambda(c_i^*) \otimes {}_{\alpha}(c_i)]$$

by lemma 2. Hence

$$\lambda(c_i^*) \# \pi(a)d^*(c_i) = \pi({}^{\alpha}a) \# \lambda(c_i^*)d^*({}_{\alpha}(c_i))$$

Theorem 1 Assume that C is a finitely generated projective k-module, then $\#(C, A, \psi^{-1}) \cong A \#_{\psi^{-1}} C^*$ as algebras.

Proof For all $a \in A, c^* \in C^*, f \in \#(C, A, \psi^{-1})$, define

$$\theta: A \#_{\psi^{-1}} C^* \to \#(C, A, \psi^{-1})$$

$$\theta(a \# c^*) = \pi(a) \# \lambda(c^*)$$

$$\vartheta: \#(C, A, \psi^{-1}) \to A \#_{\psi^{-1}} C^*$$

$$\vartheta(f) = f(c_i) \otimes c_i^*$$

We first claim that θ, ϑ are mutually inverse maps.

$$(\vartheta \circ \theta)(a \otimes c^*) = \vartheta[\theta(a \otimes c^*)] = \vartheta[\pi(a) \# \lambda(c^*)] = [\pi(a) \# \lambda(c^*)](c_i) \otimes c_i^* =$$

$$ac^*(c_i) \otimes c_i^* = a \otimes c^*$$

The last equation is inferred by ① in lemma 1.

$$(\theta \circ \vartheta(f)(c) = \theta[\vartheta(f)] = \theta(f(c_i) \otimes c_i^*)(c) = [\pi(f(c_i)) \# \lambda(c_i^*)](c) = f(c_i)c_i^*(c) = f(c)$$

Next, we claim that θ is an algebraic homomorphism.

$$\theta[(a \# c^*)(b \# d^*)] = \theta[a({}^{\alpha}b) \otimes c^*({}_{\alpha}(c_i))(d^* * c_i^*)] = \pi(a({}^{\alpha}b)) \# c^*({}_{\alpha}(c_i))\lambda(d^* * c_i^*) =$$
$$\pi(a) \# \pi({}^{\alpha}b) \# c^*({}_{\alpha}(c_i))\lambda(c_i^*) \# \lambda(d^*) =$$
$$\pi(a) \# \lambda(c^*) \# \pi(b) \# \lambda(d^*) = \theta(a \# c^*)\theta(b \# d^*)$$

The last equation is inferred by corollary 1.

Hence $\#(C, A, \psi^{-1}) \cong A \#_{\psi^{-1}} C^*$ as algebras.

2 On Ulbrich's Result

K.H.Ulbrich[8] proved the following theorem: If H is a finitely generated projective k-Hopf algebra and A is a right H-comodule algebra, then

$$A \# H^* \cong \mathrm{End}_A^H(H \otimes A)$$

In this section, we will demonstrate the same result for coalgebras.

For an entwining structure $(A, C)_\psi$, $M_A^C(\psi)$ is the category of right $(A, C)_\psi$-modules. The objects of $M_A^C(\psi)$ are right modules and right C-comodules M such that

$$\rho_M(ma) = m_{(0)}\psi(m_{(1)} \otimes a) = m_{(0)}a_\alpha \otimes m_{(1)}{}^{\alpha} \qquad \forall\, m \in M;\ a \in A$$

The morphisms in $M_A^C(\psi)$ are right A-module and right C-comodule maps.

The right A-action and right C-coaction on $C \otimes A$ are given by

$$C \otimes A \otimes A \to C \otimes A$$
$$c \otimes a \otimes b \mapsto c \otimes ab$$
$$\Delta_{C\otimes A}^r : C \otimes A \to C \otimes A \otimes C$$
$$\Delta_{C\otimes A}^r(c \otimes a) = c_{(1)} \otimes a_\alpha \otimes c_{(2)}{}^{\alpha}$$

for all $a \in A, c \in C$. It may be easily verified that $C \otimes A \in M_A^C(\psi)$.

Lemma 3 Assume that $(A, C)_\psi$ is an entwining structure, then for all $\phi \in \mathrm{End}_{-A}^{-C}(C \otimes A), c \otimes 1 \in C \otimes A$, we have

$${}_{\alpha}(c_{(1)}) \otimes {}^{\alpha}[(\varepsilon \otimes id_A)\phi(c_{(1)} \otimes 1)] = \phi(c \otimes 1)$$

Proof We write $\phi(c \otimes 1) = \sum c_i \otimes a_i$, then

$$\Delta_{C\otimes A}^r[\phi(c \otimes 1)] = \sum c_{i(1)} \otimes \psi(c_{i(2)} \otimes a_i)$$
$$(\phi \otimes C) \circ \Delta_{C\otimes A}^r(c \otimes 1) = (\phi \otimes C)(c_{(1)} \otimes \psi(c_{(2)} \otimes 1)) = (\phi \otimes C)(c_{(1)} \otimes 1 \otimes c_{(2)}) =$$
$$\phi(c_{(1)} \otimes 1) \otimes c_{(2)}$$

Hence

$$\sum c_{i(1)} \otimes \psi(c_{i(2)} \otimes a_i) = \phi(c_{(1)} \otimes 1) \otimes c_{(2)}$$

Applying $(\varepsilon \otimes id_A \otimes id_C)$ to this equality, we obtain the result that

$$(\psi \circ \phi)(c \otimes 1) = \sum \psi(c_i \otimes a_i) = (\varepsilon \otimes id_A)\phi(c_{(1)} \otimes 1) \otimes c_{(2)}$$

Thus

$${}_{\alpha}(c_{(2)}) \otimes {}^{\alpha}[(\varepsilon \otimes id_A)\phi(c_{(1)} \otimes 1)] = \phi(c \otimes 1)$$

Lemma 4 Assume that $(A, C)_\psi$ is an entwining structure. If ψ is bijective, then

$$\tilde{\omega}: \mathrm{Hom}_k(C, A) \to \mathrm{End}_{-A}^{-C}(C \otimes A)$$
$$\tilde{\omega}(f)(c \otimes a) = {}_{\alpha}(c_{(2)}) \otimes [{}^{\alpha}f(c_{(1)})]a$$

is a right C-comodule and right A-module map.

Proof For all $c \otimes a \in C \otimes A$, $f \in \mathrm{Hom}_k(C, A)$,

$$[(\tilde{\omega}(f) \otimes C) \circ \Delta_{C\otimes A}^r](c \otimes a) = (\tilde{\omega}(f) \otimes id_C)(c_{(1)} \otimes a_\gamma \otimes (c_{(2)}{}^{\gamma})) =$$
$$\tilde{\omega}(f)(c_{(1)} \otimes a_\gamma) \otimes (c_{(2)})^{\gamma} = {}_{\alpha}(c_{(2)}) \otimes {}^{\alpha}f(c_{(1)})a_\gamma \otimes c_{(3)}{}^{\gamma}$$
$$[\Delta_{C\otimes A}^r \circ \tilde{\omega}(f)](c \otimes a) = \Delta_{C\otimes A}^r[\tilde{\omega}(f)(c \otimes a)] = \Delta_{C\otimes A}^r\{{}_{\alpha}(c_{(2)}) \otimes [{}^{\alpha}f(c_{(1)})]a\} =$$
$$[{}_{\alpha}(c_{(2)})]_{(1)} \otimes \{[{}^{\alpha}f(c_{(1)})]a\}_\beta \otimes [{}_{\alpha}(c_{(2)})]_{(2)}{}^{\beta} = [{}_{\alpha}(c_{(2)})]_{(1)} \otimes [{}^{\alpha}f(c_{(1)})]_\beta a_\gamma \otimes$$

$$[{}_{\alpha}(c_{(2)})]_{(2)}{}^{\beta,\gamma} = {}_{\alpha}(c_{(2)}) \otimes [{}^{\delta,\alpha}f(c_{(1)})]_{\beta}a_{\gamma} \otimes [{}_{\delta}(c_{(3)})]^{\beta,\gamma} = {}_{\alpha}(c_{(2)}) \otimes {}^{\alpha}f(c_{(1)})a_{\gamma} \otimes c_{(3)}{}^{\gamma}$$

Hence, $\Delta^{r}_{C\otimes A} \circ \tilde{\omega}(f) = (\tilde{\omega}(f) \otimes id_C)] \circ \Delta^{r}_{C\otimes A}$, this shows that $\tilde{\omega}(f)$ is a right C-comodule. It is clearly demonstrated that $\tilde{\omega}(f)$ is a right A-linear map.

Theorem 2 Assume that $(A,C)_{\psi}$ is an entwining structure. If ψ is a bijective, then $\#(C,A,\psi^{-1}) \cong \mathrm{End}^{-C}_{-A}(C \otimes A)$ as algebras.

Proof First, we define

$$\tau: \mathrm{End}^{-C}_{-A}(C \otimes A) \to \mathrm{Hom}_k(C,A)$$
$$\tau(\phi)(c) = (\varepsilon \otimes id_A) \circ \phi(c \otimes 1)$$

for all $c \in C$, $\phi \in \mathrm{End}^{-C}_{-A}(C \otimes A)$, then

$$(\tau \circ \tilde{\omega})(f)(c) = [\tau(\tilde{\omega}(f))](c) = (\varepsilon \otimes id_A) \circ \tilde{\omega}(f)(c \otimes 1) = (\varepsilon \otimes id_A)({}_{\alpha}(c_{(1)}) \otimes {}^{\alpha}f(c_{(2)})) = \varepsilon({}_{\alpha}(c_{(1)}))\,{}^{\alpha}f(c_{(2)})) = f(c)$$

$$(\tilde{\omega} \circ \tau)(\phi)(c \otimes a) = \tilde{\omega}(\tau(\phi))(c \otimes a) = {}_{\alpha}(c_{(2)}) \otimes [{}^{\alpha}\tau(\phi)(c_{(1)})]a = {}_{\alpha}(c_{(2)}) \otimes {}^{\alpha}[(\varepsilon \otimes A) \circ \phi(c_{(1)} \otimes 1)]a = \phi(c \otimes 1)a = \phi(c \otimes a)$$

This shows that $\tilde{\omega}$ is a k-module isomorphism. Next,

$$[\tilde{\omega}(f) \circ \tilde{\omega}(g)](c \otimes a) = \tilde{\omega}(f)[\tilde{\omega}(g)(c \otimes a)] = \tilde{\omega}(f)({}_{\alpha}(c_{(2)}) \otimes {}^{\alpha}g(c_{(1)})a) = {}_{\gamma}(({}_{\alpha}(c_{(2)}))_{(2)}) \otimes [{}^{\gamma}f(({}_{\alpha}(c_{(2)}))_{(1)})][{}^{\alpha}g(c_{(1)})a] = {}_{\gamma}(({}_{\beta}(c_{(3)}))) \otimes [{}^{\gamma}f({}_{\alpha}(c_{(2)}))][{}^{\beta,\alpha}g(c_{(1)})a] = {}_{\gamma,\alpha}(c_{(3)}) \otimes [{}^{\gamma}f({}_{\beta}(c_{(2)}))][{}^{\alpha,\beta}g(c_{(1)})a] = {}_{\alpha}(c_{(3)}) \otimes {}^{\alpha}\{f[{}_{\beta}(c_{(2)})][{}^{\beta}g(c_{(1)})]\}a = {}_{\alpha}(c_{(2)}) \otimes [{}^{\alpha}(f\#g)(c_{(1)})]a = \tilde{\omega}(f\#g)(c \otimes a)$$

for all $c \otimes a \in C \otimes A$, $f,g \in \mathrm{Hom}_k(C,A)$. Hence $\tilde{\omega}$ is an algebraic isomorphism.

Corollary 2 Assume that $(A,C)_{\psi}$ is an entwining structure, C is a finitely generated projective k-module. If ψ is a bijective, then $A \#_{\psi^{-1}} C^{*} \cong \mathrm{End}^{-C}_{-A}(C \otimes A)$ as algebras.

Proof Follows immediately from lemma 2 and theorem 2.

Corollary 3[8] Let H be a Hopf algebra, and A right H-comodule algebra. If H is a finitely generated projective k-module, then $A \# H^{*} \cong \mathrm{End}^{-C}_{-A}(C \otimes A)$ as algebras.

Proof Follows immediately from example 1 and corollary 2.

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缠绕结构的 smash 积

郭广泉

(南京晓庄学院数学系，南京 210017)

摘　要　令 k 是交换环，C 是投射 k-余代数．本文讨论了具有缠绕结构 $(A,C)_{\psi}$ 的 C-余模 A 的 smash 积．当 ψ 是双射，并且 C 是有限生成投射 k-余代数时，证明了余代数的 Ulbrich 定理．

关键词　余代数；缠绕结构；smash 积

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