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Some properties and structures of solutions of the swift-Hohenberg equation

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Abstract: Stationary even periodic solutions of the Swift-Hohenberg equation are analyzed for the critical parameter $k = 1$, and it is proved that there exist periodic solutions having the same energy as the constant solution $u = 0$. For $k \leqslant 0$, some qualitative properties of the solutions are also proved.

Key words: shooting technique; Swift-Hohenberg equation; critical point; periodic solution

In this paper we shall study the Swift-Hohenberg equation

$$\frac{\partial u}{\partial t} = ku - \left(1 + \frac{\partial^2}{\partial x^2}\right)^2 u - u^3 \qquad k \in \mathbf{R} \quad (1)$$

This equation was first proposed in 1976 by Swift and Hohenberg[1] as a simple model for the Rayleigh-Bénard instability of roll waves. The Swift-Hohenberg equation has been studied a great deal, both analytically and numerically (see Refs.[2 – 10]).

In this paper we focus on stationary periodic solutions, that is on the periodic solutions $u(x)$ of the ordinary differential equation (ODE).

$$u^{\mathrm{iv}} + 2u'' + f_k(u) = 0 \quad (2)$$

where the source function f_k is given by

$$f_k(s) = (1 - k)s + s^3 \quad (3)$$

In proving the existence of periodic solutions, we use a shooting technique. We examine only even solutions of Eq.(2), and hence they suffice to construct solutions on $\mathbf{R}^+$ and continue them to $\mathbf{R}^-$ as even functions. We then study the initial value problem

$$u^{\mathrm{iv}} + 2u'' + (1 - k)u + u^3 = 0 \qquad x > 0 \quad (4)$$

$$(u, u', u'', u''')(0) = (\alpha, 0, \beta, 0) \quad (5)$$

where α and β are parameters which need to be assigned.

Our analysis will make extensive use of the fact that Eq.(2) has a first intergral, the energy identity. When we multiply (2) by $2u'$ and integrate the product, we find that each solution u satisfies the relation

$$u'u''' - \frac{1}{2}(u'')^2 + (u')^2 + F_k(u) \equiv E \quad (6)$$

where E, the energy, is a constant and

$$F_k(s) = \int_0^s f_k(t)\,\mathrm{d}t = \frac{1 - k}{2}s^2 + \frac{1}{4}s^4 \quad (7)$$

Obviously, the trivial solution $u = 0$ has energy $E \equiv 0$. When $k \leqslant 1$, being interested in branches of solutions bifurcating from $u = 0$, we shall focus on the periodic solutions having zero energy.

When we substitute (5) into the energy identity (6), and we assume $E = 0$, we find that the initial data are related through the equation

$$\beta = \pm\sqrt{2F(\alpha)} \quad (8)$$

where we have suppressed k from F_k. By symmetry, solutions come in pairs: if u is a solution, then so is $-u$. We may therefore assume without loss of generality that $u(0) = \alpha > 0$ throughout this paper and we denote the solution by $u = u(x, \alpha)$.

When $k \leqslant 0$, L.A.Peletier and W.C.Troy have proved the following result[11].

Theorem 1 If $k \leqslant 0$, then there exists no nontrivial periodic solutions of Eq.(2).

Plainly, when $k \leqslant 0$, there exists a unique constant solution $u = 0$ of (2). Since there exists no nontrivial periodic solutions of (2), we shall be concerned with the questions of existence or nonexistence of nonconstant solutions of (2) which monotonously tend to the constant solution $u = 0$ $x \to \pm\infty$. We require that

$$(u, u', u'', u''')(x) \to (0,0,0,0) \quad \text{as } x \to \pm\infty \quad (9)$$

We shall establish some qualitative properties of the solution $u(x, \alpha)$ and prove that there exists no $\alpha \in \mathbf{R}^+$ such that the initial value problem

$$u^{\mathrm{iv}} + 2u'' + (1 - k)u + u^3 = 0 \qquad x > 0 \quad (10)$$

$$(u, u', u'', u''')(0) = (\alpha, 0, \beta, 0) \quad (11)$$

where $\beta = -\sqrt{2F(\alpha)}$ has a monotone solution u which satisfies (9) at infinity.

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Define

$$x_1(\alpha) = \sup\{x > 0: u(x,\alpha) > 0 \text{ on } [0,x)\} \quad (12)$$

Theorem 2 If $k \leqslant 0$, there exists no $\alpha \in \mathbf{R}^+$ such that the initial value problem (10) and (11) has a monotone solution which tends to the constant solution $u = 0$ as $x \to \pm\infty$. More precisely, we have

(a) $x_1(\alpha) < \frac{\pi}{\sqrt{2}}$ for any $\alpha \in \mathbf{R}^+$;

(b) $\lim\limits_{\alpha \to +\infty} x_1(\alpha) = 0$;

(c) $u'(x,\alpha) < 0$ for $0 < x < x_1(\alpha)$;

(d) $u(x_1(\alpha),\alpha) = 0$, $u'(x,\alpha) < 0$ in a right-neighbourhood of $x_1(\alpha)$.

When $0 < k \leqslant 1$, to construct even periodic solutions of (2), we take $\beta = -\sqrt{2F(\alpha)}$. We shall seek a positive value of α such that the solution $u(x,\alpha)$ has the properties:

$$u''(x,\alpha) < 0,\ u(x,\alpha) > 0 \qquad 0 \leqslant x < T \quad (13)$$

$$u(T,\alpha) = 0,\ u''(T,\alpha) = 0 \quad (14)$$

for some finite $T = T(\alpha) > 0$. It follows from (13) and (14) that u is antisymmetric with respect to the point $x = T$, and so

$$u(T-x) = -u(T+x) \qquad 0 \leqslant x \leqslant T \quad (15)$$

Therefore u can be antisymmetricly continued as a solution of (2) on $[0, 2T]$. Remembering the symmetry of u around $x = 0$, we conclude that u can be continued as an even solution on $[-2T, 2T]$, and finally as a periodic solution of (2) on R with period $4T$.

Recently, Y. Tao and J. Zhang[12] have constructed the periodic solutions of (2) for the critical parameter k: $0 < k < 1$ and $1 < k < \frac{3}{2}$. In this paper we shall construct even periodic solutions of Eq. (2) for $k = 1$.

Theorem 3 Suppose that $k = 1$, then there exist even periodic solutions of Eq. (2) which have the same energy as $u = 0$. Moreover, u has countably infinite zeros and u is antisymmetric with respect to these zeros.

The plan of this paper is as follows. In section 1 we prove theorem 2 for $k \leqslant 0$. In section 2 we construct even periodic solutions for $k = 1$ and prove theorem 3 by a shooting argument.

1 Qualitative Properties of the Solution: $k \leqslant 0$

In this section, we investigate the initial value problem

$$u^{iv} + 2u'' + (1-k)u + u^3 = 0 \qquad x > 0 \quad (16)$$

$$(u, u', u'', u''')(0) = (\alpha, 0, \beta, 0) \quad (17)$$

where $\alpha > 0$ and $\beta(\alpha) = -\sqrt{2F(\alpha)}$.

We shall prove some qualitative properties of the solution $u(x,\alpha)$.

Lemma 1 Let $k \leqslant 0$. Then for any $\alpha \in \mathbf{R}^+$

$$x_1(\alpha) < \frac{\pi}{\sqrt{2}}$$

where $x_1(\alpha) \overset{\text{def}}{=} \sup\{x > 0: u(x,\alpha) > 0 \text{ on } [0,x)\}$.

Proof Let $v = \frac{u}{\alpha}$, then the problem (16) and (17) yields

$$v^{iv} + 2v'' + (1-k)v + \alpha^2 v^3 = 0 \qquad x > 0 \quad (18)$$

$$(v, v', v'', v''')(0) = \left(1, 0, -\sqrt{(1-k) + \frac{1}{2}\alpha^2}, 0\right) \quad (19)$$

For contradiction, we assume that $x_1(\alpha) \geqslant \frac{\pi}{\sqrt{2}}$. In view of the initial condition (19), one integration of (18) shows that

$$v''' + 2v' < 0 \qquad \text{on } (0, x_1(\alpha)]$$

Thus, if we set $w = v'$, and remember the initial conditions again, we obtain

$$w'' + 2w < 0 \qquad \text{on } (0, x_1(\alpha)] \quad (20)$$

$$w(0) = 0, w'(0) = -\sqrt{(1-k) + \frac{1}{2}\alpha^2} \quad (21)$$

Set $h(x) = \sqrt{\frac{1-k}{2} + \frac{1}{4}\alpha^2}\ \sin(\sqrt{2}x)$, and define the auxiliary function

$$y(x) = \frac{w(x)}{h(x)} \qquad 0 < x < \frac{\pi}{\sqrt{2}} \quad (22)$$

Then

$$y' = h^{-2}W(x) \quad (23)$$

where $W(x) = w'(x)h(x) - w(x)h'(x)$ is the Wronskian of w and h. Note that

$$W(0) = 0,\ W' = w''h - wh'' < 0 \quad \text{on } \left(0, \frac{\pi}{\sqrt{2}}\right) \quad (24)$$

since we assume that $x_1(\alpha) \geqslant \frac{\pi}{\sqrt{2}}$. Hence $W(x) < 0$, and so $y'(x) < 0$ for $0 < x < \frac{\pi}{\sqrt{2}}$. Therefore

$$y(x) < y(0) = \lim_{x \to 0^+} \frac{w'(x)}{h'(x)} = -1 \qquad 0 < x < \frac{\pi}{\sqrt{2}} \quad (25)$$

By L'Hôpital's rule, and

$$v'(x) < -\sqrt{\frac{1-k}{2} + \frac{1}{4}\alpha^2}\ \sin(\sqrt{2}x) \qquad 0 < x < \frac{\pi}{\sqrt{2}} \quad (26)$$

Integrating over $(0,x)$, this yields

$$v(x) < v^*(x) \stackrel{\text{def}}{=} 1 + \frac{1}{2}\sqrt{(1-k)+\frac{1}{2}\alpha^2}\cdot$$
$$[-1+\cos(\sqrt{2}x)] \qquad 0 < x < \frac{\pi}{\sqrt{2}} \qquad (27)$$

Note that for $k \leqslant 0$ and any $\alpha \in \mathbf{R}^+$

$$-1 < 1 - \frac{2}{\sqrt{(1-k)+\frac{1}{2}\alpha^2}} < 1$$

So, we can define

$$\tau_0(\alpha) = \frac{1}{\sqrt{2}}\arccos\left[1 - \frac{2}{\sqrt{(1-k)+\frac{1}{2}\alpha^2}}\right]$$

Then,

$$0 < \tau_0(\alpha) < \frac{\pi}{\sqrt{2}} \qquad (28)$$

We conclude from (27) and (28) that

$$v(\tau_0(\alpha)) < v^*(\tau_0(\alpha)) = 0$$

However, $v(\tau_0(\alpha)) \geqslant 0$ since we assume that $x_1(\alpha) \geqslant \frac{\pi}{\sqrt{2}}$. Thus we get a contradiction, so that $x_1(\alpha) < \frac{\pi}{\sqrt{2}}$, as asserted.

Next, we shall prove that $x_1(\alpha)$ has no positive lower bound. In fact, $u(x,\alpha) \to 0$ rapidly when α is large enough.

Lemma 2 Let $k \leqslant 1$. Then

$$\lim_{\alpha\to+\infty} x_1(\alpha) = 0$$

Proof The proof of this lemma is based on a scaling argument. It is convenient to scale the variables when α is sufficiently large. Set

$$t = \alpha^h x, v(t) = \alpha^l u(x) \qquad (29)$$

where h and l are two constants to be determined later. Then problem (16) and (17) becomes

$$v^{\text{iv}} + 2\alpha^{-2h}v'' + (1-k)\alpha^{-4h}v + \alpha^{-4h-2l}v^3 = 0 \qquad t > 0 \qquad (30)$$

$$\left.\begin{aligned} &v(0) = \alpha^{l+1} \\ &v'(0) = 0 \\ &v''(0) = -\alpha^{l-2h+1}\sqrt{(1-k)+\frac{1}{2}\alpha^2} \\ &v'''(0) = 0 \end{aligned}\right\} \qquad (31)$$

Let h and l satisfy

$$\begin{cases} 4h + 2l = 0 \\ l + 1 = 0 \end{cases}$$

which yields $h = 1/2$ and $l = -1$. Let V be the solution of the limit problem obtained from problem (30) and (31) by letting α tend to infinity:

$$V^{\text{iv}} = -V^3 \qquad t > 0 \qquad (32)$$

$$(V, V', V'', V''')(0) = \left(1, 0, -\frac{\sqrt{2}}{2}, 0\right) \qquad (33)$$

Then, since the problem (30) and (31) is a regular perturbation of problem (32) and (33), it follows that

$$(v, v', v'', v''')(t,\alpha) \to (V, V', V'', V''')(t) \quad \text{as } \alpha \to \infty \qquad (34)$$

uniformly at compact intervals. Plainly, as long as $V > 0$

$$V''' < 0,\ V'' < -\frac{\sqrt{2}}{2},\ V' < 0 \qquad (35)$$

Define

$$t_1 = \sup\{t > 0 : V > 0 \quad \text{on } (0,t)\}$$

We shall prove that

$$t_1 < 2^{\frac{3}{4}} \qquad (36)$$

In fact, integrating (32) and using the initial conditions (33), we find that

$$V'''(t) = -\int_0^t V^3(s)\,\mathrm{d}s < 0 \qquad \text{on } (0, t_1]$$

In view of the initial conditions (33) again, successive integrations yield:

$$V''(t) < -\frac{\sqrt{2}}{2} \qquad \text{on } (0, t_1] \qquad (37)$$

$$V'(t) < -\frac{\sqrt{2}}{2}t \qquad \text{on } (0, t_1] \qquad (38)$$

$$V(t) < 1 - \frac{\sqrt{2}}{4}t^2 \qquad \text{on } (0, t_1] \qquad (39)$$

We conclude from (39) that $t_1 < \infty$. In particular, we find that

$$1 - \frac{\sqrt{2}}{4}t_1^2 > V(t_1) = 0$$

which implies that (36) holds. Returning to the original variables and noticing (34) and (36), we find that if α is large enough

$$x_1(\alpha) < \frac{2^{\frac{3}{4}}}{\sqrt{\alpha}} < \frac{2}{\sqrt{\alpha}} \qquad (40)$$

which implies that $\lim_{\alpha\to+\infty} x_1(\alpha) = 0$, as asserted.

Lemma 3 Let $k \leqslant 0$. Then for any $\alpha \in \mathbf{R}^+$:

① $u(x,\alpha)$ strictly decreases on $(0, x_1(\alpha))$;

② $u(x,\alpha)$ strictly decreases in a right-neighborhood of $x_1(\alpha)$.

Proof of ① Let $v = \frac{u}{\alpha}$, then the problem (16) and (17) yields

$$v^{\text{iv}} + 2v'' + (1-k)v + \alpha^2 v^3 = 0 \qquad x > 0 \qquad (41)$$

$$(v, v', v'', v''')(0) = \left(1, 0, -\sqrt{(1-k)+\frac{1}{2}\alpha^2}, 0\right) \qquad (42)$$

To prove ①, it is necessary to prove

$$v'(x) < 0 \qquad 0 < x < x_1(\alpha) \qquad (43)$$

From the initial conditions (42) we see that for $k \leqslant 0$,

$$\left.\begin{aligned}(v''+2v)(0)&=2-\sqrt{(1-k)+\frac{1}{2}\alpha^2}<1\\(v''+2v)'(0)&=0\end{aligned}\right\}\tag{44}$$

while $0\leqslant x<x_1(\alpha)$

$$(v''+2v)''(x)=-(1-k)v-\alpha^2v^3<0\tag{45}$$

Hence, by (44),

$$(v''+2v)'<0\qquad\text{on }(0,x_1(\alpha))\tag{46}$$

and using (44)

$$v''+2v-1<0\qquad\text{on }(0,x_1(\alpha))\tag{47}$$

If now we multiply this inequality by v', then, as long as $v'<0$, we have

$$v'v''+2vv'-v'>0$$

or

$$\left[\frac{(v')^2}{2}+v^2-v\right]'>0$$

In view of the initial conditions (42), we find that as long as $v\geqslant 0$ and $v'<0$,

$$\frac{(v')^2}{2}+v^2-v>0\tag{48}$$

If $v'=0$ at some point $x_0\in(0,x_1(\alpha))$, then (48) and the continuity of the solution $v(x)$ imply that $v(x_0)\geqslant 1$, which contradicts $v(0)=1$ and $v'<0$ at $(0,x_0)$. This completes the proof of ①.

Proof of ② From ① and the definition of $x_1(\alpha)$, it is obvious that

$$u(x_1(\alpha),\alpha)=0,\ u'(x_1(\alpha),\alpha)\leqslant 0$$

Further, if $u'(x_1(\alpha),\alpha)=0$, the energy identity (6) implies that

$$u''(x_1(\alpha),\alpha)=0$$

Integrating (16) on $(0,x_1(\alpha))$, in view of the initial conditions (17), yields

$$u'''(x_1(\alpha),\alpha)=-\int_0^{x_1(\alpha)}[(1-k)u+u^3]\mathrm{d}x<0$$

So, we conclude that $u'(x,\alpha)<0$ in a right-neighborhood of $x_1(\alpha)$.

2 Periodic Solutions for $k=1$

The construction of even periodic solutions involves a detailed analysis of the initial value problem

$$u^{iv}+2u''+(1-k)u+u^3=0\qquad x>0\tag{49}$$

$$(u,u',u'',u''')(0)=(\alpha,0,-\sqrt{2F(\alpha)},0)\tag{50}$$

where $\alpha>0$ is a shooting parameter. To find periodic solutions, we need to seek a positive value of α^* such that the solution $u(x,\alpha)$ has the properties:

$$u''(x,\alpha^*)<0,\ u(x,\alpha^*)>0$$
$$0\leqslant x<T\tag{51}$$

$$u(T,\alpha^*)=0,u''(T,\alpha^*)=0\tag{52}$$

for some finite $T=T(\alpha^*)>0$. Then, u can be antisymmetricaly continued as a solution of (2) on $[0,2T]$. In view of the initial conditions (50), we conclude that u can be continued as an even solution on $[-2T,2T]$, and finally as a periodic solution of (2) on R with period $4T$. To find the above α^*, we need to study the relative position of zeros of u and u'' for the parameter α. We shall prove that u'' vanishes before u vanishes for sufficiently small $\alpha>0$, and that u vanishes before u'' vanishes for sufficiently large $\alpha>0$. Then, we use a shooting method, together with a continuity argument, to prove the existence of α^* and complete the proof of theorem 3.

Let $u(x,\alpha)$ be the solution of (49) and (50). Then, because of the requirements stated in (51), we define

$$\tau(\alpha)=\sup\{x>0:uu''<0\quad\text{on }[0,x)\}$$
$$\alpha\in\mathbf{R}^+\tag{53}$$

It is clear from the initial conditions (50) that τ is well defined. In the following lemma we prove the properties of boundedness and continuity of $\tau(\alpha)$ which are essential for our analysis.

Lemma 4 Let $0<k\leqslant 1$ and $\alpha\in\mathbf{R}^+$. Then

① $\tau(\alpha)<\infty$;

② $u(\tau(\alpha),\alpha)u''(\tau(\alpha),\alpha)=0$;

③ τ is continuous at each $\alpha\in\mathbf{R}^+$.

Proof of ① Suppose that $\tau=\infty$ for some $\alpha\in\mathbf{R}^+$. Then $u''<0$ and $u>0$ for all $x>0$. Because $u'(0)=0$, this is impossible. Thus $\tau(\alpha)<\infty$ on $\mathbf{R}^+$.

Proof of ② It follows from the definition of $\tau(\alpha)$ and the continuity of u and u''.

Proof of ③ Let $\alpha^*\in\mathbf{R}^+$, and let $\tau^*=\tau(\alpha^*)$. There are two cases to be considered.

Case 1 $u(\tau^*,\alpha^*)=0$. If $u(\tau^*,\alpha^*)=0$, then from the definition of τ we know that $u''(x,\alpha^*)<0$ on $[0,\tau^*)$. Hence, because $u'(0,\alpha^*)=0$, it follows that $u'(\tau^*,\alpha^*)<0$. Thus, we conclude from the implicit function theorem that τ is continuous at α^*.

Case 2 $u''(\tau^*,\alpha^*)=0$. If $u(\tau^*,\alpha^*)>0$, then $u''(\tau^*,\alpha^*)=0$. Hence, by the definition of τ, we have $u''(x,\alpha^*)<0$ at $[0,\tau^*)$, and therefore in view of the initial condition $u'(0,\alpha^*)=0$, we find that $u'(\tau^*,\alpha^*)<0$. Thus, we can apply the implicit function theorem again to prove that τ is continuous at α^*. This completes the proof of lemma 4.

In the following, we shall prove that u vanishes before u'' vanishes for sufficiently large $\alpha>0$. In fact, it follows from the proof of lemma 2 (see (29)-(36)).

Lemma 5 Let $0<k\leqslant 1$. Then

$$u(\tau,\alpha) = 0,\ u'(\tau,\alpha) < 0,\ u''(\tau,\alpha) < 0$$

for $\alpha \in \mathbf{R}^+$ sufficiently large.

Next, we shall prove that u'' vanishes before u vanishes for sufficiently small $\alpha > 0$. The coefficient k is found to be a critical parameter and it plays an important role in our analysis. Here we should indicate that the proof of Y. Tao and J. Zhang[12] for $0 < k < 1$ is based on a linearization argument, however, the linearization method is no longer valid for $k = 1$. For this case, we follow the method in Ref. [13] and use an auxiliary function technique.

Lemma 6 Let $k = 1$. Then

$$u(\tau,\alpha) > 0, u''(\tau,\alpha) = 0, u'''(\tau,\alpha) > 0$$

for $\alpha \in \mathbf{R}^+$ sufficiently small.

Proof It is convenient to scale the variables. Set

$$t = \sqrt{2}x,\ v(t) = \frac{1}{\alpha}u(x) \tag{54}$$

then the problem (49) and (50) yields

$$v^{\mathrm{iv}} + v'' = -\frac{\alpha^2}{4}v^3 \qquad t > 0 \tag{55}$$

$$(v, v', v'', v''')(0) = (1, 0, -\frac{\alpha}{2\sqrt{2}}, 0) \tag{56}$$

Since $v''(0) < 0$, it follows that $v'' < 0$ in a right-neighborhood of the origin. Suppose that $v'' < 0$ as long as $v > 0$. Of course, then $v' < 0$ as long as $v > 0$. We shall show that this leads to a contradiction if $\alpha \in \mathbf{R}^+$ sufficiently small.

Set $w = v'''$. Then, because $v' < 0$ as long as $v > 0$, we obtain

$$w'' + w = -\frac{3\alpha^2}{4}v^2 v' > 0$$

$$w(0) = 0$$

$$w'(0) = m(\alpha) \overset{\text{def}}{=} \frac{\alpha}{2\sqrt{2}} - \frac{\alpha^2}{4} = \frac{\alpha}{2\sqrt{2}}\left(1 - \frac{\alpha}{\sqrt{2}}\right)$$

here we take $0 < \alpha < \sqrt{2}$ and so $m(\alpha) > 0$.

As a comparison function we introduce the solution h of the initial value problem

$$h'' + h = 0,\ h(0) = 0,\ h'(0) = m(\alpha)$$

Plainly, $h(t) = m(\alpha)\sin t$. The Wronskian $W(t) = w'h - wh'$ has the properties:

$$W(0) = 0, W' > 0 \qquad \text{as long as } v > 0$$

Hence,

$$W = h^2\left(\frac{w}{h}\right)' > 0 \qquad \text{as long as } v > 0$$

Thus, since $w(t)/h(t) \to 1$ as $t \to 0^+$ by L'Hôpital's rule, it follows that as long as $v > 0$ and $h(t) > 0$

$$w(t) > h(t) = m(\alpha)\sin t$$

Here we note that $h(t) > 0$ if $0 < \alpha < \sqrt{2}$ and $t \in (0, \pi)$.

Successive integration yields

$$v''(t) > -\frac{\alpha}{2\sqrt{2}} + m(\alpha)(1 - \cos t) = \tag{57}$$

$$v'(t) > -\frac{\alpha^2}{4}t - m(\alpha)\sin t \tag{58}$$

$$v(t) > 1 - \frac{\alpha^2}{8}t^2 - m(\alpha)(1 - \cos t) \tag{59}$$

The right-hand side of (59) is decreasing, and so

$$v(t) > v(\pi) > 1 - \frac{\pi^2\alpha}{8} - \frac{\alpha}{\sqrt{2}}(1 - \frac{\alpha}{\sqrt{2}}) >$$

$$1 - 2\alpha^2 - \frac{\alpha}{\sqrt{2}} > 1 - 2\alpha - \alpha = 1 - 3\alpha > 0$$

for $0 < t < \pi$ if we take $0 < \alpha < 1/3 = \min(\sqrt{2}, 1, 1/3)$. Therefore, our assumption implies that $v'' < 0$ on $(0, \pi)$. The continuity of $v(t)$ yields $v''(\pi) \leqslant 0$. However, we see from (57) and the continuity of $v(t)$ again that

$$v''(\pi) \geqslant -\frac{\alpha^2}{4} + m(\alpha) = \frac{\alpha}{2\sqrt{2}} > 0$$

so that we have a contradiction. Thus, we must conclude that v'' vanishes before v vanishes, which means that

$$v(\tau) > 0,\ v''(\tau) = 0$$

Returning to the original variables, we find that

$$u(\tau) > 0,\ u''(\tau) = 0 \tag{60}$$

Using (60) and the energy identity (6) at $t = \tau$, one may easily assert that $u'''(\tau) > 0$. This completes the proof of lemma 6.

Now, by shooting techniques, together with lemmas 4 to 6, we can prove theorem 3.

Define the functions

$$\phi_0(\alpha) = u(\tau(\alpha), \alpha),\ \phi_2(\alpha) = u''(\tau(\alpha), \alpha) \tag{61}$$

By lemma 4 and the continuity of $u(x, \alpha)$ on the initial data, $\phi_0(\alpha)$ and $\phi_2(\alpha)$ are continuous on $\mathbf{R}^+$.

By lemma 6, there exists a small $\alpha_1 > 0$ such that

$$\phi_0(\alpha) > 0,\ \phi_2(\alpha) = 0 \qquad 0 < \alpha < \alpha_1$$

Similarly, by lemma 5, there exists a large $\alpha_2 > \alpha_1$ such that

$$\phi_2(\alpha) > 0,\ \phi_0(\alpha) = 0 \qquad \alpha > \alpha_2$$

Define the sets

$$A = \{\alpha \in \mathbf{R}^+ : u'' = 0 \text{ before } u = 0\}$$

$$B = \{\alpha \in \mathbf{R}^+ : u = 0 \text{ before } u'' = 0\}$$

where $u = u(x, \alpha)$ is the solution of the initial value problem (49) and (50).

Lemma 7 ① $A \cap B = \varnothing$;

② The sets A and B are both nonempty;

③ The sets A and B are relatively open in $\mathbf{R}^+$.

Proof of ① Part ① follows from the definition of the sets A and B.

Proof of ② Plainly, $(0,\alpha_1) \subset A, (\alpha_2, +\infty) \subset B$.

Proof of ③ Let $\alpha_0 \in A$, and let $u_0 = u(x, \alpha_0)$ and $\tau_0 = \tau(\alpha_0)$. Then

$$u''(\tau_0, \alpha_0) = 0,\ u(\tau_0, \alpha_0) > 0$$

Because u and τ depend continuously on α, there exists a small constant δ $(0 < \delta < \alpha_0)$ such that $u(\tau(\alpha), \alpha) > 0$ if $|\alpha - \alpha_0| < \delta$. According to the definition of $\tau(\alpha)$, we find that $u''(\tau(\alpha), \alpha) = 0$ if $|\alpha - \alpha_0| < \delta$. Thus the set $(\alpha_0 - \delta, \alpha_0 + \delta) \subset A$. So the set A is relatively open in $\mathbf{R}^+$.

Similarly, we can prove that the set B is relatively open in $\mathbf{R}^+$.

Proof of theorem 3 By lemma 7 and a result of Ref. [5], there exists a continuum $\Gamma \subset \mathbf{R}^+ \setminus (A \cup B)$. It follows from the definition of the sets A and B that if $\alpha \in \Gamma$,

$$u(\tau(\alpha), \alpha) = 0,\ u''(\tau(\alpha), \alpha) = 0 \tag{62}$$

Moreover, the definition of $\tau(\alpha)$ yields that if $\alpha \in \Gamma$,

$$u''(x, \alpha) < 0,\ u(x, \alpha) > 0 \qquad 0 \leqslant x < \tau(\alpha) \tag{63}$$

With the help of (62) and (63), one can easily obtain the conclusions of theorem 3.

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Swift-Hohenberg 方程解的一些性质与结构

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摘 要 讨论了 Swift-Hohenberg 方程在临界参数 $k=1$ 时的静止偶周期解，证明了其具有与非平凡解 $u=0$ 相同能量的偶周期解的存在性．对 k 是非正的情形本文也证明了方程解的一些定性性质．

关键词 打靶性；Swift-Hohenberg 方程；临界点；周期解

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