Some properties of monotone set functions defined by Choquet integral

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Abstract: In this paper, some properties of the monotone set function defined by the Choquet integral are discussed. It is shown that several important structural characteristics of the original set function, such as weak null-additivity, strong order continuity, property (s) and pseudometric generating property, etc., are preserved by the new set function. It is also shown that C-integrability assumption is inevitable for the preservations of strong order continuous and pseudometric generating property.

Key words: non-additive measure; monotone set function; Choquet integral

Given a measurable space $(X, \mathscr{F})$, a nonnegative monotone set function μ on $\mathscr{F}$ with $\mu(\phi) = 0$, and a nonnegative measurable function f, then the set function ν defined by the Choquet integral

$$\nu(E) = (\mathrm{C})\int_E f\mathrm{d}\mu \qquad \forall E \in \mathscr{F} \tag{1}$$

is also nonnegative and monotone on $\mathscr{F}$ with $\nu(\phi) = 0$[1]. This construction preserves many important structural characteristics of the original set function, such as subadditivity, null-additivity, continuity and autocontinuity[1]. In this paper, we demonstrate the preservation of other structural characteristics of μ: weak null-additivity, strong order continuity, property (s) and pseudometric generating property.

1 Preliminaries

Throughout this paper, we suppose that (X, F) is a measurable space, f is a nonnegative measurable function on (X, F), μ is a monotone set function on F with $\mu(\phi) = 0$, and $\mathbf{R}_+$ denotes $[0, +\infty)$.

The Choquet integral[2] of f on E with respect to μ, denoted by $(\mathrm{C})\int_E f\mathrm{d}\mu$, is defined as

$$(\mathrm{C})\int_E f\mathrm{d}\mu = \int_0^\infty \mu(E \cap F_\alpha)\mathrm{d}\alpha \tag{2}$$

where $F_\alpha = \{x \mid f(x) \geqslant \alpha\}$ for any $\alpha \geqslant 0$ and the integral of the right side is Lebesgue integral.

μ is said to be weakly null-additive[3] if for any $E, F \in \mathscr{F}$, $\mu(E) = \mu(F) = 0$ implies $\mu(E \cup F) = 0$; order continuity[4] if for any $\{A_n\} \subset \mathscr{F}$, $A_n \downarrow \phi$ implies $\mu(A_n) \to 0$; strong order continuity[5] if for any $\{A_n\} \subset \mathscr{F}$, $A \in \mathscr{F}$ with $\mu(A) = 0, A_n \downarrow A$ implies $\mu(A_n) \to 0$; μ is said to have property (s)[6] if for any $\{A_n\} \subset \mathscr{F}$ with $\mu(A_n) \to 0$, there exists a subsequence $\{A_{n_i}\}$ of $\{A_n\}$ such that $\mu(\overline{\lim_i} A_{n_i}) = 0$; pseudometric generating property[7], short for p.g.p., if for any $\varepsilon > 0$, there is $\delta > 0$, such that $\mu(E \cup F) < \varepsilon$ whenever $E, F \in \mathscr{F}$ and $\mu(E) \vee \mu(F) < \delta$.

2 Preservation of Structural Characteristics

Assume ν is a monotone set function defined in terms of μ by Eq.(1). Now we show that several important structural characteristics of μ are preserved in ν.

Theorem 1 If μ is weakly null-additive, then so is ν.

Proof For any $A, B \in \mathscr{F}$, if $\nu(A) = \nu(B) = 0$, that is

$$\int_0^\infty \mu(A \cap F_\alpha)\mathrm{d}\alpha = \int_0^\infty \mu(B \cap F_\alpha)\mathrm{d}\alpha = 0$$

then $\mu(A \cap F_\alpha) = 0$ m-a.e for $\alpha \in \mathbf{R}_+$ and $\mu(B \cap F_\alpha) = 0$ m-a.e for $\alpha \in \mathbf{R}_+$, where m denotes Lebesgue measure on $\mathbf{R}^1$. By the monotonicity of μ, we have

$$\mu(A \cap F_\alpha) = \mu(B \cap F_\alpha) = 0 \qquad \forall \alpha > 0$$

Therefore, it follows from the weak null-additivity of μ that

$$\nu(A \cup B) = \int_0^\infty \mu((A \cup B) \cap F_\alpha)\mathrm{d}\alpha =$$

$$\int_0^\infty \mu[(A \cap F_\alpha) \cup (B \cap F_\alpha)]\mathrm{d}\alpha =$$

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$$\int_0^\infty 0\mathrm{d}\alpha = 0$$

This means that ν is weakly null-additive.

Theorem 2 If μ has property (s), then ν also has property (s).

Proof If $\{A_n\} \subset \mathscr{F}$ and $\nu(A_n) \to 0$ $(n \to \infty)$, by Lebesgue integral theory[8], then measurable function sequence $\{\mu(A_n \cap F_\alpha)\}_n$ converges to 0 in measure $(n \to \infty)$ on R_+. Therefore, by the Riesz theorem in real analysis theory[8], there is a subsequence $\{\mu(A_{n_k} \cap F_\alpha)\}_k$ such that $\mu(A_{n_k} \cap F_\alpha) \to 0$ as $k \to \infty$ m-a.e for $\alpha \in R_+$. Noting the monotonicity of μ, we have $\mu(A_{n_k} \cap F_\alpha) \to 0$ $(k \to \infty)$ for any $\alpha > 0$. Without the loss of generality, we can assume $\mu(A_n \cap F_\alpha) \to 0$ $(n \to \infty)$ for any $\alpha > 0$. Thus, for $\alpha = \frac{1}{2}$, $\mu(A_n \cap F_{\frac{1}{2}}) \to 0$ as $n \to \infty$. By the property (s) of μ, there exists a subsequence $\{A_{n_i}^{(1)}\}$ of $\{A_n\}$, such that

$$\mu\left(\overline{\lim_{i\to\infty}}\left(A_{n_i}^{(1)} \cap F_{\frac{1}{2}}\right)\right) = 0$$

As $\mu\left(A_{n_i}^{(1)} \cap F_{\frac{1}{2^2}}\right) \to 0$ too, therefore, there exists a subsequence $\{A_{n_i}^{(2)}\}$ of $\{A_{n_i}^{(1)}\}$, such that

$$\mu\left(\overline{\lim_{i\to\infty}}\left(A_{n_i}^{(2)} \cap F_{\frac{1}{2^2}}\right)\right) = 0$$

Repeating this procedure, we can obtain a sequence $\{\varepsilon_m\}$ of subsequence of $\{A_n\}$, where $\varepsilon_m = \{A_{n_i}^{(m)}\}_i$, $m = 1, 2, \cdots$, such that $\{A_{n_i}^{(k)}\}_i \supset \{A_{n_i}^{(k+1)}\}_i$ $(k = 1, 2, \cdots)$ and

$$\mu\left(\overline{\lim_{i\to\infty}}\left(A_{n_i}^{(k)} \cap F_{\frac{1}{2^k}}\right)\right) = 0$$

Take $A_{n_i} = A_{n_i}^{(i)}$, we obtain a new subsequence $\{A_{n_i}\}$ of $\{A_n\}$. Now we show that for any $\alpha > 0$, $\mu(\overline{\lim_{i\to\infty}}(A_{n_i} \cap F_\alpha)) = 0$. For any given $\alpha > 0$, there is m_0, such that $\frac{1}{2^{m_0}} < \alpha$. Since

$$\begin{aligned}\bigcap_{k=1}^{\infty}\bigcup_{i=k}^{\infty}(A_{n_i} \cap F_\alpha) &= \bigcap_{k=m_0}^{\infty}\bigcup_{i=k}^{\infty}(A_{n_i} \cap F_\alpha) \subset \\ &\bigcap_{k=m_0}^{\infty}\bigcup_{i=k}^{\infty}\left(A_{n_i}^{(k)} \cap F_\alpha\right) \subset \\ &\bigcap_{k=m_0}^{\infty}\bigcup_{i=k}^{\infty}\left(A_{n_i}^{(m_0)} \cap F_\alpha\right) \subset \\ &\bigcap_{k=m_0}^{\infty}\bigcup_{i=k}^{\infty}\left(A_{n_i}^{(m_0)} \cap F_{\frac{1}{2^{m_0}}}\right) = \\ &\bigcap_{k=1}^{\infty}\bigcup_{i=k}^{\infty}\left(A_{n_i}^{(m_0)} \cap F_{\frac{1}{2^{m_0}}}\right) = \\ &\overline{\lim_{i\to\infty}}\left(A_{n_i}^{(m_0)} \cap F_{\frac{1}{2^{m_0}}}\right)\end{aligned}$$

Therefore, for any $\alpha > 0$, we have $0 \leqslant \mu(\overline{\lim_i}(A_{n_i} \cap F_\alpha)) \leqslant \mu\left(\overline{\lim_i}\left(A_{n_i}^{(m_0)} \cap F_{\frac{1}{2^{m_0}}}\right)\right) = 0$. Thus

$$\nu(\overline{\lim_i}A_{n_i}) = \int_0^\infty \mu(\overline{\lim_i}(A_{n_i} \cap F_\alpha))\mathrm{d}\alpha = \int_0^\infty 0\mathrm{d}\alpha = 0$$

This shows that ν has property (s).

In the following theorem 3 to theorem 5, we always suppose that f is C-integrable, i.e.,

$$(\mathrm{C})\int_X f\mathrm{d}\mu < \infty$$

Theorem 3 If μ is strongly order-continuous, then so is ν.

Proof For any $\{A_n\} \subset \mathscr{F}$ with $A_n \downarrow A$ and $\nu(A) = 0$, from

$$\int_0^\infty \mu(A \cap F_\alpha)\mathrm{d}\alpha = \nu(A) = 0$$

and the monotonicity of μ, we know that $\forall \alpha > 0$, $\mu(A \cap F_\alpha) = 0$. It follows from the strongly order-continuous of μ that $\mu(A_n \cap F_\alpha) \downarrow 0$ $(n \to \infty)$, $\forall \alpha > 0$.

On the other hand,

$$\int_0^\infty \mu(A_1 \cap F_\alpha)\mathrm{d}\alpha \leqslant \int_0^\infty \mu(F_\alpha)\mathrm{d}\alpha = (\mathrm{C})\int_X f\mathrm{d}\mu < \infty$$

By using the Lebesgue convergence theorem[8], we obtain

$$\nu(A_n) = \int_0^\infty \mu(A_n \cap F_\alpha)\mathrm{d}\alpha \to 0 \qquad n \to \infty$$

This means that ν is strongly order-continuous.

In a similar way, we can obtain the following results.

Theorem 4 If μ is order-continuous, then so is ν.

Theorem 5 If μ has the p.g.p., then ν also has the p.g.p.

Proof For any given $\varepsilon > 0$, noting that f is C-integrable, i.e., $\int_0^\infty \mu(F_\alpha)\mathrm{d}\alpha = \nu(X) < \infty$, therefore there exist a and b $(0 < a < b)$, such that

$$\int_0^a \mu(F_\alpha)\mathrm{d}\alpha + \int_b^\infty \mu(F_\alpha)\mathrm{d}\alpha < \frac{\varepsilon}{2}$$

Since μ has the p.g.p., there exists $\delta > 0$, such that for any $E, F \in \mathscr{F}$

$$\mu(E) \vee \mu(F) < \delta \Rightarrow \mu(E \cup F) < \frac{\varepsilon}{2(b-a)} \tag{3}$$

For this given ε, we take $\delta_1 = a\delta$. Now we show that for any $A, B \in \mathscr{F}$, $\nu(A) \vee \nu(B) < \delta_1$ implies $\nu(A \cup B) < \varepsilon$. In fact, we have

$$\int_0^a \mu(A \cap F_\alpha)\,\mathrm{d}\alpha \vee \int_0^a \mu(B \cap F_\alpha)\,\mathrm{d}\alpha < \delta_1 = a\delta$$

By using the monotonicity of μ, we have $\mu(A \cap F_\alpha) \vee \mu(B \cap F_\alpha) < \delta$ for any $\alpha \in [a, \infty)$. Therefore, it follows from Eq.(3) that $\mu((A \cup B) \cap F_\alpha) < \frac{\varepsilon}{2(b-a)}$ for any $\alpha \in [a, \infty)$.

Consequently,

$$\int_a^b \mu\Big((A \cup B) \cap F_\alpha\Big)\,\mathrm{d}\alpha < \frac{\varepsilon}{2}$$

Thus we have

$$\begin{aligned}\nu(A \cup B) &= \int_0^a \mu\Big((A \cup B) \cap F_\alpha\Big)\,\mathrm{d}\alpha + \\ &\int_b^\infty \mu\Big((A \cup B) \cap F_\alpha\Big)\,\mathrm{d}\alpha + \\ &\int_a^b \mu\Big((A \cup B) \cap F_\alpha\Big)\,\mathrm{d}\alpha \leqslant \\ &\int_0^a \mu(F_\alpha)\mathrm{d}\alpha + \int_b^\infty \mu(F_\alpha)\mathrm{d}\alpha + \\ &\int_a^b \mu\Big((A \cup B) \cap F_\alpha\Big)\,\mathrm{d}\alpha < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

This shows that ν has the p.g.p.

Remark Observe that theorem 3 to theorem 5 are based on the assumption that f is C-integrable, i.e., ν is finite. The following examples show that the conclusions in theorem 3 to theorem 5 may not be true when the assumption is abandoned.

Example 1 Let $X = \{1, 2, \cdots\}$ and $F = P(X)$. Let μ_1 be a set function defined on F by

$$\mu_1(E) = \begin{cases} 0 & E = \varnothing \\ \max\{\frac{1}{i} \mid i \in E\} & E \neq \varnothing \end{cases}$$

It is easy to verify that μ_1 is a monotone set function and it is strongly order-continuous (hence order-continuous).

We take $f(x) = x$ $(x \in X)$ and let ν_1 be the set function defined by Choquet integral of f with respect to μ_1. Then, by calculating, we have $\nu_1(X) = \infty$, therefore f is not C-integrable with respect to μ_1. Now we show that the set function ν_1 is not order-continuous and hence not strongly order-continuous. In fact, we take $E_n = \{n, n+1, \cdots\}$, then $E_n \downarrow \phi$, but

$$\nu_1(E_n) = \int_0^\infty \mu_1(E_n \cap F_\alpha)\,\mathrm{d}\alpha \geqslant \int_0^n \mu_1(E_n \cap F_\alpha)\,\mathrm{d}\alpha = 1$$

Example 2 Let $X_1 = \{1, 3, \cdots\}$, $X_2 = \{2, 4, \cdots\}$, $X = X_1 \cup X_2$, and $F = P(X)$. Define set function μ_2 on F as follows:

$$\mu_2(E) = \begin{cases} 0 & E = \varnothing \\ \max\{\frac{1}{i^2} \mid i \in E\} & E \subset X_1 \text{ or } E \subset X_2 \\ \max\{\frac{1}{i} \mid i \in E\} & \text{otherwise} \end{cases}$$

It is not too difficult to verify that μ_2 is a monotone set function with the p.g.p.

We take a measurable function $f(x) = x$ $(x \in X)$ and let ν_2 be the set function defined by Choquet integral of f with respect to μ_2. Then f is not C-integrable with respect to μ_2. In the following we show that the monotone set function ν_2 has not the p.g.p. In fact, if we take $A_n = \{2n\}$, $B_n = \{2n+1\}$, then

$$\lim_{n\to\infty} \nu_2(A_n) = \lim_{n\to\infty} \int_0^\infty \mu_2(A_n \cap F_\alpha)\,\mathrm{d}\alpha = \lim_{n\to\infty} \int_0^{2n} \frac{1}{(2n)^2}\mathrm{d}\alpha = \lim_{n\to\infty} \frac{1}{2n} = 0$$

Similarly, $\nu_2(B_n) = \frac{1}{2n+1} \to 0$ $(n \to \infty)$. However,

$$\lim_{n\to\infty} \nu_2(A_n \cup B_n) \geqslant \lim_{n\to\infty} \int_0^{2n} \mu_2((A_n \cup B_n) \cap F_\alpha)\,\mathrm{d}\alpha = \lim_{n\to\infty} \int_0^{2n} \frac{1}{2n}\mathrm{d}\alpha = 1$$

This shows that ν_2 has not the p.g.p.

3 Concluding Remarks

In this paper we have proved that several structural characteristics of a non-additive set function, such as weak null-additivity, strong order continuity, property (s) and pseudometric generating property, are preserved in the new set function defined by the Choquet integral. The structural characteristics concerned in the paper are introduced and discussed in Refs.[3,5,6,9], separately. Since these concepts play important roles in non-additive measure theory in Refs.[3,5,6,7,9,10], the results obtained in this paper are noteworthy.

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Choquet 积分定义的单调集函数的几个遗传性质

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摘　要　讨论了 Choquet 积分定义的单调集函数的几个遗传性质.证明了 Choquet 积分定义的新的单调集函数遗传了原来集函数的几个重要的结构特性,如弱零可加性、强序连续性、性质(S)和伪距离生成性质等.最后通过 2 个例子说明了当被积函数不是 C 可积时,强序连续性和伪距离生成性质将不再被保留.

关键词　非可加测度; 单调集函数; Choquet 积分

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