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Finite element method of the eigenvalues of Sturm-Liouville's problem

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Abstract: This paper considers the finite element method of the approximate value of eigenvalues of Sturm-Liouville's problem. The proof of our main result is based on the variational method. Linear interpolating functions are made by interpolation method, the problem of the approximate value of eigenvalues becomes the calculation of eigenvlaues of a matrix. Then the finite element method of the approximate value of the eigenvalues is obtained, and accuracy of $(n-1)$-th approximate value is estimated by n-th approximate value. When n is increased, the accuracy of eigenvalue λ_k is increased. When n is appropriately selected, the accuracy of λ_k we need is obtained. This finite element method is significant both in applications and in theory.

Key words: Sturm-Liouville's problem; eigenvalue; eigenfunction; finite element method

1 Main Result

Let $(a, b) \subset \mathbf{R}$ be a bound interval. We consider the approximate value of the eigenvalues of Sturn-Liouville's problem

$$\left.\begin{aligned} &-[p(x)y']' + q(x)y = \lambda s(x)y \qquad x \in (a,b) \\ &y(a) = y(b) = 0 \end{aligned}\right\} \tag{1}$$

where $p(x) \in C^1([a, b])$, $q(x), s(x) \in C([a, b])$, such that $p(x) > 0$, $q(x) \geqslant 0, s(x) > 0$, $x \in [a, b]$.

The estimates for bound of $(n+1)$-th eigenvalue of problem (1) are well known[1-5]. The approximate value of the eigenvalues of a similar problem (1) is calculated by Galerkin method[6]. The computational method is simpler, and accuracy is higher, but a lot of integral computation is required, and the condition is stronger, i.e. $p(x) \geqslant \mu_1 > 0$ and $s(x) \geqslant \mu_2 > 0$. In this paper, the approximate value of the eigenvalues of Sturn-Liouville's problem (1) is calculated by the finite element method. The condition in Ref. [6] is weakened, i.e. it is only required that $p(x) > 0$ and $s(x) > 0$. This finite element method is interesting and significant both in applications and in theory.

Our main result is based on the variational method. First of all, a theorem is proved. Secondly, linear interpolating functions are made by interpolation method. At last, the problem of the approximate values of eigenvalues becomes the calculation of eigenvlaues of a matrix, the finite element method of the approximate value of the eigenvalues is obtained immediately.

Let $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n \leqslant \cdots$ denote the successive eigenvalues for (1). Let $C^1([a, b])$ denote the set of functions having first derivative continuous in $[a, b]$. Let $C_0^1([a, b])$ denote the set of functions in $C^1([a, b])$ with compact support in $[a, b]$. Let $L_0^2([a, b])$ denote the set of measurable functions that are 2-integrable in $[a, b]$ with compact support in $[a, b]$. The set of $C_0^1([a, b])$ is the dense subset in $L_0^2([a, b])$.

Also let

$$D(y) = \frac{1}{2}\int_a^b [p(x)(y')^2 + q(x)y^2]\mathrm{d}x \tag{2}$$

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$$E(y) = \frac{1}{2}\int_a^b s(x)y^2 \mathrm{d}x \tag{3}$$

$$\left.\begin{aligned} J(y) &= \frac{D(y)}{E(y)} \\ y(a) &= y(b) = 0 \end{aligned}\right\} \tag{4}$$

Let $u \neq 0$ be a function, and $u \in L_0^2([a,b]), u' \in L_0^2([a,b])$. If u is a critical function of functional $J(y)$, then $J(u)$ is called a critical value.

Theorem A $C_0^1([a,b])$ function u is a critical function of (4) if and only if u is an eigenfunction of (1). Consequently the critical value $J(u)$ is equal to eigenvalue λ of (1), i.e. $J(u) = \lambda$.

Proof Let $u \neq 0$, $u \in C_0^1([a,b])$. Let u be a critical function of $J(y)$. As $y \neq 0$, $E(y) > 0$, then we have $E(u) > 0$. Finding the variation of $J(y)$, we have

$$\delta J(y) = \frac{1}{E^2(y)}[E(y)\delta D(y) - D(y)\delta E(y)] = \frac{1}{E(y)}[\delta D(y) - J(y)\delta E(y)] \tag{5}$$

Replacing y in (5) by u, let $J(u) = \lambda$, we obtain

$$\delta J(u) = \frac{1}{E(u)}[\delta D(u) - J(u)\delta E(u)] = \frac{1}{E(u)}[\delta D(u) - \lambda\delta E(u)] = 0$$

Since $u \neq 0$, we get

$$\delta D(u) - \lambda\delta E(u) = 0 \tag{6}$$

By the variation for (2),(3) and integration by parts, for any $\delta v \in C_0^1([a,b])$, we have

$$\delta D(u) = \int_a^b [p(x)u'\delta v' + q(x)u\delta v]\mathrm{d}x = \int_a^b \{-[p(x)u']' + q(x)u\}\delta v \mathrm{d}x \tag{7}$$

$$\delta E(u) = \int_a^b s(x)u\delta v \mathrm{d}x \tag{8}$$

By (6), (7), and (8), we obtain

$$\int_a^b \{-[p(x)u']' + q(x)u - \lambda s(x)u\}\delta v \mathrm{d}x = 0 \qquad \forall\, \delta v \in C_0^1([a,b]) \tag{9}$$

Therefore $-[p(x)u']' + q(x)u = \lambda s(x)u$ satisfying the condition $u(a) = u(b) = 0$, i.e. a critical function u in (4) is an eigenfunction of (1). Similarly, the eigenfunction u of (1) is a critical function in (4). Then the critical value $J(u)$ is equal to the eigenvalue λ of (1), i.e. $J(u) = \lambda$. The set of $C_0^1([a,b])$ is the dense subset in $L_0^2([a,b])$, consequently the theorem also holds for any $u \neq 0$, $u \in L_0^2([a,b])$ and $u' \in L_0^2([a,b])$.

2 Finite Element Method

By the theorem, the approximate values of the eigenvalues of (1) are calculated by the finite element method. Let $a = x_0 < x_1 < x_2 < \cdots < x_n = b, y(x_i) = y_i, i = 0,1,2,\cdots,n$, $\varphi_i(x)$ are given by

$$\varphi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & x \in [x_{i-1}, x_i) \\ 1 & x = x_i \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & x \in (x_i, x_{i+1}] \\ 0 & x \notin [x_{i-1}, x_{i+1}] \end{cases}$$

Let

$$y_n(x) = \sum_{i=1}^{n-1} y_i\varphi_i(x) \tag{10}$$

such that ① $y_n(x_i) = y_i$; ② $y_n(x)$ is continuous in [a,b]; ③ $y_n'(x)$ has some discontinuous point, but is 2-integrable.

Replacing y in (2) and (3) by $y_n(x)$ in (10), we have

$$D(y_n(x)) = \frac{1}{2}\int_a^b \left[p(x)\Big(\sum_{i=1}^{n-1} y_i\varphi_i'\Big)^2 + q(x)\Big(\sum_{i=1}^{n-1} y_i\varphi_i\Big)^2\right]\mathrm{d}x =$$

$$\frac{1}{2}\sum_{i,j=1}^{n-1}\int_a^b [p(x)\varphi_i'\varphi_j' y_i y_j + q(x)\varphi_i\varphi_j y_i y_j]\mathrm{d}x \tag{11}$$

$$E[y_n(x)] = \frac{1}{2}\sum_{i,j=1}^{n-1}\int_a^b s(x)\varphi_i\varphi_j y_i y_j \mathrm{d}x \tag{12}$$

Let

$$a_{ij} = \int_a^b [p(x)\varphi_i'(x)\varphi_j'(x) + q(x)\varphi_i\varphi_j]\mathrm{d}x,\ a_{ij} = a_{ji} \qquad i,j = 1,2,\cdots,n-1 \tag{13}$$

$$b_{ij} = \int_a^b s(x)\varphi_i(x)\varphi_j(x)\mathrm{d}x,\ b_{ij} = b_{ji} \qquad i,j = 1,2,\cdots,n-1 \tag{14}$$

By (11), (12), (13), and (14), we obtain

$$D[y_n(x)] = D(y_1, y_2, \cdots, y_{n-1}) = \frac{1}{2}\sum_{i,j=1}^{n-1} a_{ij} y_i y_j = \frac{1}{2}\boldsymbol{y}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{y},\ a_{ij} = a_{ji} \tag{15}$$

$$E(y_n(x)) = E(y_1, y_2, \cdots, y_{n-1}) = \frac{1}{2}\sum_{i,j=1}^{n-1} b_{ij} y_i y_j = \frac{1}{2}\boldsymbol{y}^{\mathrm{T}}\boldsymbol{B}\boldsymbol{y},\ b_{ij} = b_{ji} \tag{16}$$

where $\boldsymbol{A} = [a_{ij}]$ is usually a positive definite or a positive semidefinite matrix; $\boldsymbol{B} = [b_{ij}]$ is a positive definite matrix. Consequently the problem (4) becomes the problem of critical values of a multivariate function, i.e.

$$J(y_1, y_2, \cdots, y_{n-1}) = \frac{D(y_1, y_2, \cdots, y_{n-1})}{E(y_1, y_2, \cdots, y_{n-1})} \tag{17}$$

As $(y_1, y_2, \cdots, y_{n-1}) \neq 0$, $(y_1, y_2, \cdots, y_{n-1})$ is a critical point of functional J, we have

$$\frac{\partial}{\partial y_i} J(y_1, y_2, \cdots, y_{n-1}) = 0 \qquad i = 1,2,\cdots,n-1 \tag{18}$$

The value corresponding to functional J is called the critical value, i.e. $J(y_1, y_2, \cdots, y_{n-1}) = \lambda$. Since

$$\frac{\partial}{\partial y_i} J = \frac{\partial}{\partial y_i}\left(\frac{D}{E}\right) = \frac{1}{E^2}\left(E\frac{\partial D}{\partial y_i} - D\frac{\partial E}{\partial y_i}\right) = \frac{1}{E}\left(\frac{\partial D}{\partial y_i} - J\frac{\partial E}{\partial y_i}\right) = \frac{1}{E}\left(\frac{\partial D}{\partial y_i} - \lambda\frac{\partial E}{\partial y_i}\right)$$

by $(y_1, y_2, \cdots, y_{n-1}) \neq 0$ and (18), we obtain

$$\frac{\partial D}{\partial y_i} - \lambda\frac{\partial E}{\partial y_i} = 0 \qquad i = 1,2,\cdots,n-1 \tag{19}$$

Since

$$\frac{\partial}{\partial y_i} D(y_1, y_2, \cdots, y_{n-1}) = \frac{1}{2}\sum_{j=1}^{n-1} a_{ij} y_j,\ \frac{\partial}{\partial y_i} E(y_1, y_2, \cdots, y_{n-1}) = \frac{1}{2}\sum_{j=1}^{n-1} b_{ij} y_j$$

so that by (19)

$$\sum_{j=1}^{n-1} a_{ij} y_j = \lambda\sum_{j=1}^{n-1} b_{ij} y_j \tag{20}$$

By (20), we get

$$\boldsymbol{A}\boldsymbol{y} = \lambda\boldsymbol{B}\boldsymbol{y} \tag{21}$$

Consequently the problem (1) becomes the algebraic problem (21) of an eigenvalue.

The interval $[a, b]$ is divided into n subintervals, the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ in Eq.(21) are calculated. Corresponding function J is

$$D(y) = \frac{1}{2}\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} [p(x)(y')^2 + q(x)y^2]\mathrm{d}x = \frac{1}{2}\sum_{i=1}^{n} I_i \tag{22}$$

On the i-th interval, the linear interpolating function is

$$y^{(i)}(x) = \tau_{i1}(x)y_{i-1} + \tau_{i2}(x)y_i \tag{23}$$

where $\tau_{i1}(x) = \dfrac{x_i - x}{x_i - x_{i-1}}$, $\tau_{i2}(x) = \dfrac{x - x_{i-1}}{x_i - x_{i-1}}$.

Replacing y in I_i by $y^{(i)}(x)$ in Eq.(23), computing I_i, we obtain

$$I_i = \int_{x_{i-1}}^{x_i} p(x)[(\tau_{i1}')^2 y_{i-1}^2 + 2\tau_{i1}'\tau_{i2}' y_{i-1} y_i + (\tau_{i2}')^2 y_i^2]\mathrm{d}x +$$

$$\int_{x_{i-1}}^{x_i} q(x)[(\tau_{i1})^2 y_{i-1}^2 + 2\tau_{i1}\tau_{i2} y_{i-1} y_i + (\tau_{i2})^2 y_i^2]\mathrm{d}x = I_{i1} + I_{i2} \tag{24}$$

By $\tau'_{i1}(x) = \dfrac{-1}{x_i - x_{i-1}}$ and $\tau'_{i2}(x) = \dfrac{1}{x_i - x_{i-1}}$, computing I_{i1} in (24), we have

$$I_{i1} = \frac{y_{i-1}^2 - 2y_{i-1}y_i + y_i^2}{(x_i - x_{i-1})^2}\int_{x_{i-1}}^{x_i} p(x)\mathrm{d}x \tag{25}$$

By taking $p(x) = p_i$ in $[x_{i-1}, x_i]$ and (25), we get

$$I_{i1} = \frac{p_i}{x_i - x_{i-1}}(y_{i-1}^2 - 2y_{i-1}y_i + y_i^2) \tag{26}$$

By $\tau_{i1}(x) = \dfrac{x_i - x}{x_i - x_{i-1}}$ and $\tau_{i2}(x) = \dfrac{x - x_{i-1}}{x_i - x_{i-1}}$, computing I_{i2} in (24), we get

$$I_{i2} = \int_{x_{i-1}}^{x_i} \frac{q(x)}{(x_i - x_{i-1})^2}[(x_i - x)^2 y_{i-1}^2 + 2(x_i - x)(x - x_{i-1})y_{i-1}y_i + (x - x_{i-1})^2 y_i^2]\mathrm{d}x \tag{27}$$

By taking $q(x) = q_i$ in $[x_{i-1}, x_i]$ and (27), we obtain

$$I_{i2} = \frac{q_i}{3}(x_i - x_{i-1})(y_{i-1}^2 + y_{i-1}y_i + y_i^2) \tag{28}$$

Using (26), (28), and (24), we have

$$I_i = I_{i1} + I_{i2} = \frac{p_i}{x_i - x_{i-1}}(y_{i-1}^2 - 2y_{i-1}y_i + y_i^2) + \frac{q_i}{3}(x_i - x_{i-1})(y_{i-1}^2 + y_{i-1}y_i + y_i^2) \tag{29}$$

Combining (22) and (20) yields

$$D[y_n(x)] = \frac{1}{2}\sum_{i=1}^{n} I_i = \frac{1}{2}\sum_{i=1}^{n}\Big[\frac{p_i}{x_i - x_{i-1}}(y_{i-1}^2 - 2y_{i-1}y_i + y_i^2) + \frac{q_i}{3}(x_i - x_{i-1})(y_{i-1}^2 + y_{i-1}y_i + y_i^2)\Big] = \frac{1}{2}\boldsymbol{y}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{y}$$

where $\boldsymbol{y} = \{y_1, y_2, \cdots, y_{n-1}\}^{\mathrm{T}}$, $\boldsymbol{A} = [a_{ij}]$ is a three-diagonal matrix. The main diagonal element of matrix $\boldsymbol{A}$ is

$$a_{ii} = \frac{p_i}{h_i} + \frac{p_{i+1}}{h_{i+1}} + \frac{1}{3}(q_i h_i + q_{i+1}h_{i+1}) \qquad i = 1,2,\cdots,n-1 \tag{30}$$

The secondary diagonal element of matrix $\boldsymbol{A}$ is

$$a_{i,i+1} = a_{i+1,i} = -\frac{p_{i+1}}{h_{i+1}} + \frac{1}{6}q_{i+1}h_{i+1} \qquad i = 1,2,\cdots,n-2 \tag{31}$$

Similarly, we have

$$E[y_n(x)] = \frac{1}{2}\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} s(x)y^2\mathrm{d}x \tag{32}$$

By taking $s(x) = s_i$ in $[x_{i-1}, x_i]$ and (32), we get

$$E[y_n(x)] = \frac{1}{2}\sum_{i=1}^{n}\frac{s_i}{3}(x_i - x_{i-1})(y_{i-1}^2 + y_{i-1}y_i + y_i^2) = \frac{1}{2}\boldsymbol{y}^{\mathrm{T}}\boldsymbol{B}\boldsymbol{y}$$

where $\boldsymbol{B} = [b_{ij}]$ is a three-diagonal positive definite matrix. The main diagonal element of matrix $\boldsymbol{B}$ is

$$b_{ii} = \frac{1}{3}(s_i h_i + s_{i+1}h_{i+1}) \qquad i = 1,2,\cdots,n-1 \tag{33}$$

The secondary diagonal element of matrix $\boldsymbol{B}$ is

$$b_{i,i+1} = b_{i+1,i} = \frac{1}{6}s_{i+1}h_{i+1} \qquad i = 1,2,\cdots,n-2 \tag{34}$$

where $h_i = x_i - x_{i-1}$; $p_i = p(x_i)$; $q_i = q(x_i)$; $s_i = s(x_i)$.

Using (30), (31), (33), and (34), the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are calculated. The eigenvalues of $\boldsymbol{A}\boldsymbol{y} = \lambda\boldsymbol{B}\boldsymbol{y}$ can be calculated. Consequently we obtain the approximate value of eigenvalues of the problem (1).

3 Two Examples

Example 1

$$\begin{cases} -y'' = \lambda y & x \in (0,1) \\ y(0) = y(1) = 0 \end{cases}$$

Let $p(x) = 1, q(x) = 0, s(x) = 1$ satisfying the condition of (1). The interval $[0, 1]$ is divided into n same subintervals. By taking $h_i = h = \frac{1}{n}$, using (30), (31), (33), and (34), the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are calculated, we have

$$a_{ii} = 2n,\ b_{ii} = \frac{2n}{3} \qquad i = 1,2,\cdots,n-1$$

$$a_{i,i+1} = a_{i+1,i} = -n,\ b_{i,i+1} = b_{i+1,i} = \frac{1}{6n} \qquad i = 1,2,\cdots,n-2$$

i.e.

$$\left.\begin{aligned}&-ny_{i-1} + 2ny_i - ny_{i+1} = \lambda\left(\frac{y_{i-1}}{6n} + \frac{2}{3n}y_i + \frac{y_{i+1}}{6n}\right) \qquad i = 1,2,\cdots,n-1\\&y_0 = y_n = 0\end{aligned}\right\} \tag{35}$$

By taking $n = 25$ in (35), we get the approximate values of eigenvalues $\lambda_1 = 9.8826$, $\lambda_2 = 39.6867$, $\lambda_3 = 89.8834$ and $\lambda_4 = 161.266$, in comparison with the accurate value of the eigenvalues of the original equation, λ_1 and λ_2 each have two effective digits. λ_3 and λ_4 each have one effective digit.

Example 2

$$\begin{cases}-[(x+1)y']' + xy = \lambda(x^2+1)y \qquad x \in (0,1)\\ y(0) = y(1) = 0\end{cases}$$

Let $p(x) = x+1, q(x) = x, s(x) = x^2+1$ satisfying the condition of (1). The interval $[0,1]$ is divided into n same subintervals. By taking $h_i = h = 1/n$, using (30), (31), (33), and (34), the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are calculated, we obtain

$$a_{ii} = 2n + 2i + 1 + \frac{2i+1}{3n^2},\ b_{ii} = \frac{1}{3n^3}[2n^2 + i^2 + (i+1)^2] \qquad i = 1,2,\cdots,n-1$$

$$a_{i,i+1} = a_{i+1,i} = -n - i - 1 + \frac{i+1}{6n^2},\ b_{i,i+1} = b_{i+1,i} = \frac{n^2+i^2}{6n^3} \qquad i = 1,2,\cdots,n-2$$

i.e.

$$\left.\begin{aligned}&\left(\frac{i}{6n^2} - n - i\right)y_{i-1} + \left(2n + 2i + 1 + \frac{2i+1}{3n^2}\right)y_i + \left(\frac{i+1}{6n^2} - n - i - 1\right)y_{i+1} =\\&\quad\lambda\left(\frac{n^2+(i-1)^2}{6n^3}y_{i-1} + \frac{2n^2+i^2+(i+1)^2}{3n^3}y_i + \frac{n^2+i^2}{6n^3}y_{i+1}\right) \qquad i = 1,2,\cdots,n-1\\&y_0 = y_n = 0\end{aligned}\right\} \tag{36}$$

By taking $n = 20$ in (36), we get approximate values of eigenvalues $\lambda_1 = 11.9905$, $\lambda_2 = 46.267$, $\lambda_3 = 104.729$ and $\lambda_4 = 189.369$, in comparison with the accurate value of the eigenvalues in the original equation, λ_1 and λ_2 each have two effective digits. λ_3 and λ_4 each have one effective digit.

By taking $n = 60$ in example 1, we get approximate values of eigenvalues $\lambda_1 = 9.86985$, $\lambda_2 = 39.4904$, $\lambda_3 = 88.9879$ and $\lambda_4 = 158.747$. λ_1 and λ_2 each have three effective digits. λ_3 and λ_4 each have two effective digits. Therefore the accuracy of $(n-1)$-th approximate value is estimated by n-th approximate value. When n is increased, the accuracy of eigenvalue λ_k is increased. When n is appropriately selected, the accuracy of λ_k we need is obtained.

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Sturm-Liouville 问题特征值的有限元方法

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摘　要　本文构建了计算 Sturm-Liouville 问题特征值的有限元方法.主要结果的证明运用了变分法.通过构造适当的线性插值函数,将微分方程特征值的近似计算问题离散化为矩阵特征值计算问题.从而获得了微分方程特征值的近似值的有限元方法,而且可以用第 n 次近似值来估计第 $n-1$ 次的近似值的精确度. 随着 n 的增大,特征值 λ_k 的精确度逐步提高,只要适当选取 n,就可以求得所要求精确度的特征值的近似值,这个算法具有广泛的实用价值和理论价值.

关键词　Sturm-Liouville 问题; 特征值; 特征函数; 有限元方法
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