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Asymptotic upper bounds for wheel: complete graph Ramsey numbers

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Abstract: It is shown that $r(W_m, K_n) \leqslant (1+o(1))C_1\left(\frac{n}{\log n}\right)^{\frac{2m-2}{m-2}}$ for fixed even $m \geqslant 4$ and $n \to \infty$, and $r(W_m, K_n) \leqslant (1+o(1))C_2\left(\frac{n^{\frac{2m}{m+1}}}{\log n}\right)^{\frac{m+1}{m-1}}$ for fixed odd $m \geqslant 5$ and $n \to \infty$, where $C_1 = C_1(m) > 0$ and $C_2 = C_2(m) > 0$, in particular, $C_2 = 12$ if $m = 5$. It is obtained by the analytic method and using the function $f_m(x) = \int_0^1 \frac{(1-t)^{\frac{1}{m}}\mathrm{d}t}{m+(x-m)t}$, $x \geqslant 0$, $m \geqslant 1$ on the base of the asymptotic upper bounds for $r(C_m, K_n)$ which were given by Caro, et al. Also, $c\left(\frac{n}{\log n}\right)^{\frac{5}{2}} \leqslant r(K_4, K_n) \leqslant (1+o(1))\frac{n^3}{(\log n)^2}$ (as $n \to \infty$). Moreover, we give $r(K_k + C_m, K_n) \leqslant (1+o(1))C_5(m)\left(\frac{n}{\log n}\right)^{k+\frac{m}{m-2}}$ for fixed even $m \geqslant 4$ and $r(K_k + C_m, K_n) \leqslant (1+o(1))C_6(m)\left(\frac{n^{\frac{2+(k+1)(m-1)}{2+k(m-1)}}}{\log n}\right)^{k+\frac{2}{m-1}}$ for fixed odd $m \geqslant 3$ (as $n \to \infty$).

Key words: Ramsey numbers; wheels; independent number; complete graphs

Let H be a graph without isolates. The Ramsey number $r(H, K_n)$ is the smallest integer N such that each graph on N vertices that fails to contain H as a subgraph has an independence number of at least n. Denote by C_m the cycle of length m and by W_m the wheel of order $m+1$, where $W_m = K_1 + C_m$. The "join" of graghs K and H, denoted by $K+H$, is the graph obtained by starting with vertex disjoint copies of K and H and adding uv to the edge set for every $u \in V(K)$ and $v \in V(H)$. Chvátal[1] proved $r(T_m, K_n) = 1+(m-1)(n-1)$, where T_m is any tree of the order m. A generalization[2] of a result of Spencer[3] shows that for the connected graph G, $r(G, K_n)$ is linearly bounded if and only if G is a tree, and the cases where G contains a cycle are complicated (see lemma 2 in this paper). In this paper, we give asymptotic bounds for $r(W_m, K_n)$ and some related Ramsey numbers.

Caro, et al[4] have proved that when m is even and for $m \geqslant 4$, as $n \to \infty$,

$$r(C_m, K_n) \leqslant C_1(m)\left(\frac{n}{\log n}\right)^{\frac{m}{m-2}} \tag{1}$$

where $C_1(m)$ and henceforth $C_i(m)$ are positive constants. For $m = 3$, the order of magnitude of $r(C_3, K_n)$, viz. $\frac{n^2}{\log n}$, but the different values in different places, was determined by Ajtai, et al.[5] and Kim[6]. For odd $m \geqslant 3$, Li and Zang[7] proved that as $n \to \infty$,

$$r(C_m, K_n) \leqslant C_2(m)\left(\frac{n^{\frac{m+1}{2}}}{\log n}\right)^{\frac{2}{m-1}} \tag{2}$$

in particular, $C_2(m) = 6$ if $m = 5$.

1 Upper Bounds for $r(W_m, K_n)$

We will use the function $f_m(x)$[8, 9] as

$$f_m(x) = \int_0^1 \frac{(1-t)^{\frac{1}{m}}\mathrm{d}t}{m+(x-m)t} \qquad x \geqslant 0,\ m \geqslant 1$$

which plays a central role. Li and Rousseau[8] originally got its property by a complex method. Later Li, et al. got it by an improved method in Ref.[9]. Clearly $f_m(x)$ is a decreasing function. Since $(1-t)^{\frac{1}{m}} \geqslant (1-t)$ for $0 \leqslant t \leqslant 1$ and $m \geqslant 1$, a simple calculation gives

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$$f_m(x) \geqslant \int_0^1 \frac{(1-t)\mathrm{d}t}{m+(x-m)t} = \frac{x\log\frac{x}{m}-(x-m)}{(x-m)^2} > \frac{\log\frac{x}{m}-1}{x} \qquad x>m \tag{3}$$

With $\frac{x}{m}=1+u$, the last inequality is equivalent to $(1+2u)\log(1+u)>u$, which holds for all $u>0$ since $\log(1+u)>\frac{u}{1+u}$ (see Ref.[4]).

Also, see Ref.[9],

$$f_m(x) \geqslant \frac{1}{1+x} \qquad x \geqslant m \tag{4}$$

Throughout the remainder of this paper, we shall let G_v stand for the subgraph of a graph G induced by the neighborhood of v.

Lemma 1 Let G be a graph with N vertices and average degree d. If for any vertex v of G, the average degree of G_v is at most a, then $\alpha(G) \geqslant Nf_{a+1}(d)$.

The proof of lemma 1, see Ref.[9].

By combining the upper bounds of $r(C_m, K_n)$ and lemma 1, we shall manage to get the following upper bound for $r(W_m, K_n)$.

Theorem 1 ① Let $m \geqslant 4$ be any fixed even integer. Then for sufficiently large n, we have

$$r(W_m, K_n) \leqslant (1+o(1))C_3(m)\left(\frac{n}{\log n}\right)^{\frac{2m-2}{m-2}}$$

where $C_3(m)=\frac{C_1(m)(m-2)}{2}$.

② Let $m \geqslant 3$ be any fixed odd integer. Then for sufficiently large n, we have

$$r(W_m, K_n) \leqslant (1+o(1))C_4(m)\left(\frac{n^{\frac{2m}{m+1}}}{\log n}\right)^{\frac{m+1}{m-1}}$$

where $C_4(m)=\frac{C_2(m)(m-1)}{2}$. In particular, $C_4(m)=12$ if $m=5$.

Proof Let G be a graph of order $N=r(W_m, K_n)-1$ such that G contains no W_m and $\alpha(G) \leqslant n-1$. Then for each vertex v of G, we have ① the degree of v is at most $r(C_m, K_n)-1$, and ② the maximum degree and therefore the average degree of G_v is at most $r(P_{m-1}, K_n)-1$, where P_{m-1} is the path of length $m-1$.

Thus, it follows from lemma 1 that

$$n > \alpha(G) \geqslant Nf_a(r(C_m, K_n)-1) \geqslant Nf_a(r(C_m, K_n)) \tag{5}$$

where $a=r(P_{m-1}, K_n)=(m-2)(n-1)+1<r(C_m, K_n)$.

① When m is even and $m \geqslant 4$, then, by (1), for sufficiently large n,

$$n \geqslant Nf_a\left(C_1\left(\frac{n}{\log n}\right)^{\frac{m}{m-2}}\right)$$

By (3),

$$n \geqslant N\frac{\log\frac{C_1\left(\frac{n}{\log n}\right)^{\frac{m}{m-2}}}{(m-2)(n-1)+1}-1}{C_1\left(\frac{n}{\log n}\right)^{\frac{m}{m-2}}} \geqslant N\frac{\log\frac{C_1\left(\frac{n}{\log n}\right)^{\frac{m}{m-2}}}{mn}-1}{C_1\left(\frac{n}{\log n}\right)^{\frac{m}{m-2}}}$$

where $C_1=C_1(m)$ in (1). Hence, as $n\to\infty$,

$$C_1 n\left(\frac{n}{\log n}\right)^{\frac{m}{m-2}} \geqslant N\left((1-o(1))\frac{2}{m-2}\log n\right)$$

it yields

$$N \leqslant \frac{C_1(m-2)n}{2(1-o(1))\log n}\left(\frac{n}{\log n}\right)^{\frac{m}{m-2}}$$

So, as $n\to\infty$,

$$r(W_m, K_n)=N+1 \leqslant (1+o(1))C_3(m)\left(\frac{n}{\log n}\right)^{\frac{2m-2}{m-2}}$$

where $C_3(m)=\frac{C_1(m)(m-2)}{2}$.

② The case where m is odd and $m \geqslant 3$ can be proved on the base of (2), by the same analytic calculation in ①; here we omit it, completing the proof.

2 Asymptotic Bounds for $r(K_4, K_n)$

Lemma 2[2] For any fixed integer $m \geqslant 3$, constants $\delta>0$ and $\alpha \geqslant 0$, if F is a graph on m vertices and G is a graph on n vertices with $e(G) \geqslant \frac{(\delta-o(1))n^2}{(\log n)^{\alpha}}$ as $n\to\infty$, then there exists a constant $c=c(m,\delta)>0$ such that

$$r(F, G) \geqslant (c-o(1))\left(\frac{n}{(\log n)^{\alpha+1}}\right)^{\frac{e(F)-1}{m-2}}$$

Lemma 2 is a generalization of Spencer's following result.

Lemma 3[3] For any fixed integer $m \geqslant 3$, there is a constant $c>0$ such that

$$r(C_m, K_n) \geqslant c\left(\frac{n}{\log n}\right)^{\frac{m-1}{m-2}}$$

Lemma 4 Let $m \geqslant 3$ be a fixed integer. Then, as $n\to\infty$,

$$r(W_m, K_n) \leqslant (1+o(1))(m-2)r(C_m, K_n)\frac{n}{\log n} \quad (6)$$

Proof By (3) and lemma 3, we can obtain lemma 4; here we omit it.

Li, et al[9] proved that as $n\to\infty$, $r(K_k, K_n) \leqslant \frac{(1+o(1))n^{k-1}}{(\log n)^{k-2}}$. Note that $K_4 = K_1 + C_3$. By lemma 2 and lemma 4, we can get the following corollary.

Corollary As $n\to\infty$, there exists a constant $c > 0$ such that

$$c\left(\frac{n}{\log n}\right)^{\frac{5}{2}} \leqslant r(K_4, K_n) \leqslant (1+o(1))\frac{n^3}{(\log n)^2}$$

3 Upper Bounds for $r(K_k + C_m, K_n)$

For $m=3$, as $n\to\infty$, $r(K_k + C_3, K_n) = r(K_{k+3}, K_n) \leqslant \frac{(1+o(1))n^{k+2}}{(\log n)^{k+1}}$.

Theorem 2 Let m and k be any two fixed integers and $m \geqslant 3$. Then, as $n\to\infty$,

① When m is even,

$$r(K_k + C_m, K_n) \leqslant (1+o(1))C_5(m) \cdot \left(\frac{n}{\log n}\right)^{k+\frac{m}{m-2}}$$

② When m is odd,

$$r(K_k + C_m, K_n) \leqslant (1+o(1))C_6(m) \cdot \left(\frac{n^{\frac{2+(k+1)(m-1)}{2+k(m-1)}}}{\log n}\right)^{k+\frac{2}{m-1}}$$

where $C_5(m) = (m-2)C_1(m)$, $C_6(m) = (m-2)\cdot C_2(m)$.

Proof of ① When m is even, it is proved by induction on k.

For $k=0$, by (1),

$$r(C_m, K_n) \leqslant C_1(m)\left(\frac{n}{\log n}\right)^{\frac{m}{m-2}}$$

For $k=1$, by (6),

$$r(W_m, K_n) \leqslant (1+o(1))(m-2)C_1(m)\left(\frac{n}{\log n}\right)^{1+\frac{m}{m-2}}$$

Our statement follows.

Suppose that the statement holds for $0, 1, 2, \cdots, k$. We proceed to the induction step. Let $r(k, m; n)$ denote $r(K_k + C_m, K_n)$ and let G be a graph of order $N = r(k+1, m; n) - 1$ such that G contains no $K_{k+1} + C_m$ and that $\alpha(G) \leqslant n-1$. Then for each vertex v of G, we have ① the degree of v is at most $r(k, m; n) - 1$, and ② the maximum degree and therefore the average degree of G_v is at most $r(k-1, m; n) - 1$. Thus, by lemma 1,

$$n > \alpha(G) \geqslant Nf_a(r(k, m; n) - 1) \geqslant Nf_a(r(k, m; n)) \quad (7)$$

where $a = r(k-1, m; n)$. Now let ε be an arbitrary number with $0 < \varepsilon < 1$. Then, by (3) we know that there exists an $M > 0$ such that $f_a(x) > \frac{(1-\varepsilon)\log\frac{x}{a}}{x}$ whenever $\frac{x}{a} > M$. We decompose the set of large natural numbers into n' and n'' such that

$$\frac{r(k, m; n')}{r(k-1, m; n')} > (n')^{1-\varepsilon}$$

and

$$\frac{r(k, m; n'')}{r(k-1, m; n'')} \leqslant (n'')^{1-\varepsilon}$$

Thus $\log\frac{r(k, m; n')}{r(k-1, m; n')} \geqslant (1-\varepsilon)\log n'$. Without loss of generality, we may suppose that for all n', $(n')^{(1-\varepsilon)} > M$. So by (7), we have

$$n' > Nf_a(r(k, m; n')) \geqslant (1-\varepsilon)N\frac{\log\frac{r(k, m; n')}{a}}{r(k, m; n')} \geqslant \frac{(1-\varepsilon)^2 N\log n'}{r(k, m; n')}$$

where $a = r(k-1, m; n')$. Hence $N \leqslant \frac{n'}{(1-\varepsilon)^2} \cdot \frac{r(k, m; n')}{\log n'}$, and the desired inequality for $r(k+1, m; n')$ follows from the induction hypothesis on $r(k, m; n')$. By (4), $f_a(x) \geqslant \frac{1}{1+x}$, if $x \geqslant a$, we get

$$n'' > Nf_a(r(k, m; n'')) \geqslant \frac{N}{1+r(k, m; n'')}$$

where $a = r(k-1, m; n'') < r(k, m; n'')$. Hence,

$$N \leqslant n''(1+r(k, m; n'')) \leqslant n''(1+(n'')^{1-\varepsilon}r(k-1, m; n''))$$

The desired inequality for $r(k+1, m; n'')$ follows from the induction hypothesis on $r(k-1, m; n'')$ since $(n'')^{2-\varepsilon} < \left(\frac{n''}{\log n''}\right)^2$ for sufficiently large n''. Thus we complete the proof of the case where m is even.

The case where m is odd can be proved by a similar method; here we omit it.

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对于轮和完全图的 Ramsey 函数的渐近上界

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摘要: 给出了当 n 趋向于无穷时, 对于不小于 4 的偶数 m, 有 $r(W_m, K_n) \leqslant (1+o(1))C_1 \cdot \left(\frac{n}{\log n}\right)^{\frac{2m-2}{m-2}}$; 对于不小于 5 的奇数 m, 有 $r(W_m, K_n) \leqslant (1+o(1))C_2\left(\frac{n^{\frac{2m}{m+1}}}{\log n}\right)^{\frac{m+1}{m-1}}$. 这里 $C_1 = C_1(m) > 0$, $C_2 = C_2(m) > 0$. 特别地, $C_2(5) = 12$. 该定理是在 Caro 等给出的 $r(C_m, K_n)$ 的渐近上界的基础上利用函数 $f_m(x) = \int_0^1 \frac{(1-t)^{\frac{1}{m}}\mathrm{d}t}{m+(x-m)t}$ 得到的. 当 n 趋向于无穷时, $c\left(\frac{n}{\log n}\right)^{\frac{5}{2}} \leqslant r(K_4, K_n) \leqslant (1+o(1))\frac{n^3}{(\log n)^2}$. 本文还给出了 $r(K_k + C_m, K_n)$ 的渐近上界.

关键词: Ramsey 数; 轮; 独立数; 完全图
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