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Generalized super-Virsoro algebras and their Verma modules

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Abstract: For any additive subgroup M of a field F and $\alpha \in F$ such that $2\alpha \in M$, there are two classes of generalized super-Virasoro algebras denoted by $\mathrm{SVir}[M, \alpha]$ and $\widetilde{\mathrm{SVir}}[M, \alpha]$ by Su and Zhao. The latter is in fact a trivial extension of the former. In this paper, based on the discussion on isomorphisms, the Verma modules of $\mathrm{Svir}[M, \alpha]$ are studied, and the irreducibility of these modules are obtained.

Key words: Lie algebra; module; isomorphism

Virasoro algebra and the related algebraic structures play an important role in many areas of mathematics and mathematical physics, and theories about their structures and representations have been well developed. The high ranking Virasoro algebras and high ranking super-Virasoro algebras were studied in Refs. [1, 2], respectively. The generalized Virasoro algebras and generalized super-Virasoro algebras were studied in Ref.[3]. Important modules, such as Harish-Chandra modules, modules of intermediate series, indecomposable modules, highest weight modules and Verma modules of some of these algebras are also extensively studied[3-9]. In particular, the automorphisms of generalized Virasoro algebra were determined in Ref. [3]. The irreducibility of their Verma modules was determined in Ref. [4].

The main purpose of this paper is to study the Verma modules of generalized super-Virasoro algebras, the algebras introduced by Su and Zhao[3]. In section 1, some presentations of these algebras are given. Similar to Ref. [3], we determine their automorphisms and give sufficient and necessary conditions under which two such algebras are isomorphic. In section 2, we define the Verma modules of generalized super-Virasoro algebras and determine the irreducibility of these modules.

1 Generalized Super-Virasoro Algebras

Let M be an additive subgroup of field F, $\alpha \in F$ such that $2\alpha \in M$. By Ref. [3], there are two classes of generalized super-Virasoro algebras with respect to M, α, which are denoted by $\mathrm{SVir}[M, \alpha]$ and $\widetilde{\mathrm{SVir}}[M, \alpha]$, respectively. They both have basis $\{L_x \mid x \in M\} \cup \{G_v \mid v \in \alpha + M\} \cup \{c\}$. $\mathrm{SVir}[M, \alpha]$ is a Lie superalgebra with commutative relationships:

$$\left.\begin{aligned}
&[L_x, L_y] = (y-x)L_{x+y} + \delta_{x+y,0}\frac{x^3-x}{12}c\\
&[L_x, c] = [c, G_v] = 0\\
&[L_x, G_v] = \left(v-\frac{x}{2}\right)G_{x+v}
\end{aligned}\right\} \tag{1}$$

$$[G_{v_1}, G_{v_2}] = 2L_{v_1+v_2} - \delta_{v_1+v_2,0}\frac{1}{3}({v_1}^2-\frac{1}{4})c \tag{2}$$

$\widetilde{\mathrm{SVir}}[M, \alpha]$ is a Lie superalgebra with the commutative relationships (1) and

$$[G_{v_1}, G_{v_2}] = \delta_{v_1+v_2,0}c \tag{3}$$

Clearly, $\widetilde{\mathrm{SVir}}[M, \alpha]$ is a trivial extension of $\mathrm{Vir}[M]$ modulo the center Fc, so we will mainly discuss $\mathrm{SVir}[M, \alpha]$. All results in this section can be obtained in the ways similar to those used in Ref. [3]. So, the proofs of the results in this section are all omitted.

For the automorphism group of $\mathrm{SVir}[M, \alpha]$, we have the following theorem.

Theorem 1 ① For any $\chi \in \mathrm{Hom}(I, F^*)$, the linear transformation φ_χ over $\mathrm{SVir}[M, \alpha]$ determined by

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$$\varphi_\chi(c)=c,\ \varphi_\chi(E_x)=\chi(x)E_x \qquad x\in I$$

is an automorphism, where $E_x=L_x$ if $x\in M$, $E_x=G_x$ if $x\in M'$.

② For any $a\in\{a\in F\mid aM=M\}$, the linear transformation φ'_a over SVir$[M,\ \alpha]$ determined by

$$c\rightarrow ac$$

$$L_x\rightarrow a^{-1}L_{ax}-d_{x,0}\frac{a-a^{-1}}{24}c$$

$$G_v\rightarrow pG_{av}$$

where $p^2=a^{-1}$ is an automorphism of SVir$[M,\ \alpha]$.

③ Aut (SVir$[M,\ \alpha]$) $=\varphi_{\mathrm{Hom}(M,\ F^*)}\rtimes\varphi_{S(I)}$.

Remark 1 The theorem generalized the results of theorem 2. 3 in Ref. [3]. Note that the form of the automorphisms φ'_a in ② is not the same as those in Ref. [3]. In fact, verification shows that φ'_a in Ref. [3] should be determined by the following correspondences:

$$c\rightarrow ac$$

$$L_x\rightarrow a^{-1}L_{ax}-\delta_{x,0}\frac{a-a^{-1}}{24}c$$

By Ref. [3], if M, N are two additive subgroups of F, then the generalized Virasoro algebras Vir$[M]$ and Vir$[N]$ are isomorphic if and only if there exists $a\in F^*$ such that $aM=N$. In the case of generalized super-Virasoro algebras, the similar result holds. In fact, we have the following theorem.

Theorem 2 Let M, N be two additive subgroups of F. For $\alpha,\beta\in F$ such that $2\alpha\in M$, $2\beta\in N$, let $M'=M+\alpha$, $N'=N+\beta$, $I=M\cup M'$, $J=N\cup N'$. Then, SVir$[M,\ \alpha]$ and SVir$[N,\ \beta]$ are isomorphic if and only if there exists $a\in F^*$ such that $aM=N$ and $aI=J$.

Corollary 1 For any $a\in M^*:=M\backslash\{0\}$, SVir$\left[M,\ \frac{a}{2}\right]$ has a subalgebra SVir$\left[aZ,\ \frac{a}{2}\right]$, which is isomorphic to SVir$\left[Z,\ \frac{1}{2}\right]$ and the mapping θ: SVir $\left[Z,\ \frac{1}{2}\right]\rightarrow$SVir$\left[aZ,\ \frac{a}{2}\right]$:

$$c\rightarrow ac$$

$$L_i\rightarrow a^{-1}L_{ai}-\delta_{i,0}\frac{a-a^{-1}}{24}c$$

$$G_{\frac{1}{2}+j}\rightarrow pG_{\frac{a}{2}+ja}$$

with $p^2=a^{-1}$, is an isomorphism between SVir $\left[Z,\ \frac{1}{2}\right]$ and SVir $\left[aZ,\ \frac{a}{2}\right]$.

Corollary 2 Let $\alpha\in F$ with $\alpha\notin M$ and $2\alpha\in M,\beta\in M$, then SVir$[M,\ \alpha]$ is isomorphic to the subalgebra of SVir$[M,\ \beta]$ generated by $\{L_{2u}\mid u\in M\}\cup\{G_{2v+2\alpha}\mid v\in M\}$ and there exists an injective homomorphism θ of SVir$[M,\ \alpha]$ such that

$$c\rightarrow 2c$$

$$L_u\rightarrow\frac{1}{2}L_{2u}-\delta_{u,0}\frac{1}{16}c$$

$$G_{v+\alpha}\rightarrow pG_{2v+2\alpha}$$

for some $p\in F$ and $p^2=\frac{1}{2}$.

In the sequel, we will mainly consider the case of $\alpha\notin M$, $2\alpha\in M$. The results in the case of $\alpha\in M$ can be then obtained according to corollary 2.

2 Verma Modules and Their Irridučibility

Clearly, $I=M\cup M'$ is an additive subgroup of F. Let "$>$" be a total order on I compatible with the addition, i.e., $a+c>b+c\Leftrightarrow a>b$. Let $I_+=\{i\in I\mid i>0\}$, $I_-=-I_+$ then $I=I_+\cup\{0\}\cup I_-$. Then, SVir$[M,\ \alpha]$ has triangular decomposition:

$$\mathrm{SVir}[M,\ \alpha]=\mathrm{SVir}_-\oplus\mathrm{SVir}_0\oplus\mathrm{SVir}_+$$

when $\alpha\notin M$, it has triangular decomposition:

$$\mathrm{SVir}[M,\ \alpha]=\mathrm{SVir}_-\oplus\mathrm{SVir}_0\oplus\mathrm{SVir}_+\oplus FG_0$$

when $\alpha \in M$, where $\mathrm{SVir}_{-} = \oplus_{i \in I_-} FE_i$, $\mathrm{SVir}_0 = FL_0 \oplus Fc$, $\mathrm{SVir}_{+} = \oplus_{i \in I_+} FE_i$. So,

$$U(\mathrm{SVir}[M, \alpha]) = U(\mathrm{SVir}_-)U(\mathrm{SVir}_0)U(\mathrm{SVir}_+) \qquad \alpha \notin M$$

$$U(\mathrm{SVir}[M, \alpha]) = U(\mathrm{SVir}_-)U(\mathrm{SVir}_0)U(FG_0)U(\mathrm{SVir}_+) \qquad \alpha \in M$$

For convenience, denote by $(E_i)_{i \in I}$ the basis of the base vector space of $\mathrm{SVir}[M, \alpha]$ such that

$$E_i = L_i (i \in M), \ E_i = G_i (i \in M')$$

Let $\bar{E} = \{(i_1, i_2, \cdots, i_r) \mid \forall r \in Z_+ \cup \{0\}, i_1 \leqslant i_2 \leqslant \cdots \leqslant i_r$, and $i_p < i_{p+1}$ if $i_p, i_{p+1} \in M'\}$. So, by PBW's theorem, the universal enveloping algebra of SVir $[M, \alpha]$ has basis:

$$\{E_{i_1} \cdots E_{i_r}, \ E_{i_1} \cdots E_{i_r} c \mid (i_1, \cdots i_r) \in \bar{E}\}$$

Let $\dot{c}$, $h \in F$, V_h a one-dimensional vector space over F with basis v_h. Define the actions of L_0 and c on V_h by $L_0 . v_h = hv_h$, $c. v_h = \dot{c} v_h$. Then V_h becomes one-dimensional SVir_0-module. Define the action of $b = \mathrm{SVir}_0 + \mathrm{SVir}_+$ over V_h by $\mathrm{SVir}_+ . v_h = 0$, then V_h can be viewed as a b-module.

Definition 1 The induced module

$$\mathrm{Ind}_b^{\mathrm{SVir}[M, \alpha]} V_h = U(\mathrm{SVir}[M, \alpha]) \otimes_{U(b)} V_h$$

over $\mathrm{SVir}[M, \alpha]$ is called a Verma module of $\mathrm{SVir}[M, \alpha]$, and is denoted by $M(\dot{c}, h)$.

It is easy to see that c acts on $M(\dot{c}, h)$ as scalar $\dot{c}$. We denote the weight space $\{v \in M(\dot{c}, h) \mid L_0 . v = \lambda v, c. v = \dot{c} v\}$ with weight λ of $M(\dot{c}, h)$ by V_λ.

Lemma 1 If $\alpha \in M$, then $G_0 . v_h = 0$.

Proof If $\alpha \in M$, $I = M$, we define the Z_2-graded of $M(\dot{c}, h)$ by $M(\dot{c}, h) = M_{\bar{0}} \oplus M_{\bar{1}}$, where $M_{\bar{0}}$ is the space spanned by $\{E_{-i_1} \cdots E_{-i_r} v_h \mid \forall r \geqslant 0, \ \forall j, \ i_j \in M_+, \ (i_1, \cdots, i_r) \in \bar{E}$, and the number of E_{-i_j} of form G_{-i_j} is even$\}$, $M_{\bar{1}}$ is the space spanned by $\{E_{-i_1} \cdots E_{-i_r} v_h \mid \forall r \geqslant 0, \ \forall j, \ i_j \in M_+, \ (i_1, \cdots i_r) \in \bar{E}$, and the number of E_{-i_j} of form G_{-i_j} is odd$\}$. Denote $M_{\bar{0}}^{\lambda} = M_{\bar{0}} \cap V_\lambda$. $M_{\bar{1}}^{\lambda} = M_{\bar{1}} \cap V_\lambda$. Since $L_0 . (G_0 . v_h) = G_0 . (L_0 . v_h) = h(G_0 . v_h)$, we have $G_0 . v_h \in M_{\bar{1}}^h$. Since $V_h = M_{\bar{0}}^h \oplus M_{\bar{1}}^h$ is a one-dimensional space and $M_{\bar{0}}^h = Fv_h$, so, $G_0 . v_h \in M_{\bar{1}}^h = \{0\}$.

For any $x \in I_+$, let $M_x(\dot{c}, h)$ be the submodule of $M(\dot{c}, h)$ over $\mathrm{SVir}[2xZ, x]$ generated by the highest fixed weight generator of $M(\dot{c}, h)$. By corollary 1, we have the following corollary.

Corollary 3 If we define the action of $\mathrm{SVir}\left[Z, \frac{1}{2}\right]$ on $M_x(\dot{c}, h)$ by

$$c. v_h = 2xc. v_h, \ G_{\frac{1}{2}+j} . v_h = pG_x . v_h$$

$$L_i . v_h = \left[(2x)^{-1} L_{2xi} - \delta_{i,0} \frac{2x - (2x)^{-1}}{24} c\right] . v_h$$

$M_x(\dot{c}, h)$ becomes a $\mathrm{SVir}\left[Z, \frac{1}{2}\right]$-module and as a $\mathrm{SVir}\left[Z, \frac{1}{2}\right]$-module, $M_x(\dot{c}, h)$ is isomorphic to $M\left(2x\dot{c}, (2x)^{-1} h - \frac{2x - (2x)^{-1}}{24} \dot{c}\right)$, where $p^2 = (2x)^{-1}$.

Remark 2 From Ref. [4], it is known that for any total order "$>$" of I, either

$$\text{for any } x \in I_+, \ \#\{y \in I \mid 0 < y < x\} = \infty$$

or

$$\exists a \in I, \text{ such that } \#\{y \in I \mid 0 < y < a\} = \varnothing$$

The order "$>$" is called dense in the first case, discrete in the second case.

Theorem 3 If the order "$>$" of I is dense, then Verma module $M(\dot{c}, h)$ of $\mathrm{SVir}[M, \alpha]$ is irreducible if and only if $(\dot{c}, h) \neq (0, 0)$.

Proof Let v_h be a fixed highest weight generator in $M(\dot{c}, h)$ of weight h. For each $m \in N$, we set

$$V_m := \sum_{\substack{0 \leqslant r \leqslant m \\ (i_1, i_2, \cdots, i_r) \in \bar{E} \\ i_1, i_2, \cdots, i_r \in I_+}} FE_{-i_1} \cdots E_{-i_r} v_h$$

It can be easily proved that $E_i V_m \subseteq V_m$ for any $i \in I_+$.

We prove the theorem in two cases.

Case 1 $\alpha \notin M$.

Let $u_0 \neq 0$ be any given weight vector in $M(\dot{c}, h)$, we prove that

$$E_{-x}v_h \in U(\mathrm{SVir}[M, \alpha])u_0 \text{ for any } x \in I_+$$

Claim 1 There exists a weight vector $u \in U(\mathrm{SVir}[M, \alpha])u_0$ with weight λ such that, for some $r \in N$,

$$u \equiv L_{-\varepsilon_r}L_{-\varepsilon_{r-1}}\cdots L_{-\varepsilon_1}v_h(\mathrm{mod}V_{r-1})$$

where $\varepsilon_j \in M_+ (j=1, \cdots, r)$ and $(-\varepsilon_r, \cdots, -\varepsilon_1) \in \bar{E}$.

In fact, it is clear that $u_0 \in V_r$ for some $r \in N$. If $r=0$, we have $u_0 \in Fv_h$, let $u=v_h$(hence $\lambda = h$), then claim 1 holds. If $r=1$, without loss of generality, we suppose that the coefficient is 1 and $u_0 = L_{-\varepsilon_1}v_h$ or $u_0 = G_{-\varepsilon_1}v_h$. For the first case, let $u=u_0$(hence $\lambda = h-\varepsilon_1$), then claim 1 holds. For the second case, let $u = G_{\varepsilon_1-\varepsilon_2}G_{-\varepsilon_1}v_h = 2L_{-\varepsilon_2}v_h$, where $\varepsilon_2 \in M_+$ and $\varepsilon_1 > \varepsilon_2$, (hence $\lambda = h-\varepsilon_2$), then claim 1 holds. If $r>1$, we rewrite

$$u_0 \equiv \sum_{\substack{(i_1, \cdots, i_r) \in \bar{E} \\ i_1, \cdots, i_r \in I_+}} a_{\underline{i}}E_{-i_1}\cdots E_{-i_r}v_h(\mathrm{mod}V_{r-1})$$

where $\underline{i} = (i_1, \cdots, i_r)$. Let $\bar{I} = \{(i_1, \cdots, i_r) \mid a_{\underline{i}} \neq 0\}$. By the assumption, $\bar{I} \neq \emptyset$. For any $\underline{i}$, $\underline{i}' \in \bar{I}$, define

$$\underline{i} \succ \underline{i}' \Leftrightarrow \exists \text{ integer } s,\ 1 \leqslant s \leqslant r, \text{ such that } i_s > i_s' \text{ and } i_t = i_t' \text{ for } t > s$$

Let $\underline{j} = (j_1, \cdots, j_r) \in$ be the maximal element of $\bar{I}$. Because "$\succ$" is a dense order, we can always find some $\varepsilon_1 \in M_+$ such that $\varepsilon_1 < j_1$ and $\{x \in I \mid j_r - \varepsilon_1 < x < j_r\} \cap \{i_s \mid 1 \leqslant s \leqslant r,\ \underline{i} \in \bar{I} = \emptyset$. Then,

$$u_1 = E_{j_r-\varepsilon_1}u_0 \equiv \sum_{\substack{(\varepsilon_1, i_1, \cdots, i_{r-1}) \in \bar{E} \\ \varepsilon_1 \in M_+ \\ i_1, \cdots, i_r \in I_+}} a_{\underline{i}}^{(1)}L_{-\varepsilon_1}E_{-i_1^{(1)}}\cdots E_{-i_{r-1}}^{(1)}v_h(\mathrm{mod}\ V_{r-1}) \tag{4}$$

where $\bar{I}^{(1)} = \{(\varepsilon_1, i_1^{(1)}, \cdots, i_{r-1}^{(1)}) \mid a_{\underline{i}}^{(1)} \neq 0\} \neq \emptyset$. In this step, only those satisfying $i_r = j_r$ in $\{a_{\underline{i}}E_{-i_1}\cdots E_{-i_r} \mid a_{\underline{i}} \neq 0\}$ produce the terms of the sum in (4), the rests are acted to V_{r-1}.

We denote $\underline{j}^{(1)} = (\varepsilon_1, j_1^{(1)}, \cdots, j_{r-1}^{(1)})$ the maximal element of $\bar{I}^{(1)}$. Let $\varepsilon_2 \in M_+$, such that $\varepsilon_2 < \varepsilon_1$ and $\{x \in I \mid j_{r-1} - \varepsilon_2 < x < j_{r-1}\} \cap \{i_s^{(1)} \mid 1 \leqslant s \leqslant r-1,\ \underline{i}^{(1)} = (\varepsilon_1, i_1^{(1)}, \cdots, i_{r-1}^{(1)}) \in \bar{I}^{(1)} = \emptyset$. Then,

$$u_2 = E_{j_{r-1}-\varepsilon_2}u_1 \equiv \sum_{\substack{(\varepsilon_2, \varepsilon_1, i_1, \cdots, i_{r-2}) \in \bar{E} \\ \varepsilon_2, \varepsilon_1, i_1, \cdots, i_{r-2} \in I_+}} a_{\underline{i}}^{(2)}L_{-\varepsilon_2}L_{-\varepsilon_1}E_{-i_1}^{(2)}\cdots E_{-i_{r-2}}^{(2)}v_h(\mathrm{mod}\ V_{r-1}) \tag{5}$$

Similarly, in this step, only those satisfying $i_{r-1} = j_{r-1}$ in $\{a_{\underline{i}}^{(1)}L_{-\varepsilon_1}E_{-i_1^{(1)}}\cdots E_{-i_{r-1}^{(1)}} \mid a_i^{(1)} \neq 0\}$ produce the terms of the sum in (5), the rest are acted to V_{r-1}.

Recursively, we can repeat this process step by step and obtain the following equation:

$$u = u_r \equiv aL_{-\varepsilon_r}\cdots L_{-\varepsilon_1}v_h(\mathrm{mod}\ V_{r-1})$$

for some nonzero $a \in F$. Without loss of generality, we can set $a=1$. **So, the claim holds.**

Claim 2 There exists some $\varepsilon \in I_+$ such that $E_{-\varepsilon}v_h \in U(\mathrm{SVir}[M, \alpha])u_0$. Furthermore $E_{-x}v_h \in U(\mathrm{SVir}[M, \alpha])u_0$ for all of $x \in B(\varepsilon)$, where $B(\varepsilon)$ is the semigroup generated by the set $\{y \in I_+ \mid y \leqslant \varepsilon\}$.

In fact, according to claim 1, there exists some weight vector $u \in U(\mathrm{SVir}[M, \alpha])u_0$ such that

$$u = L_{-\varepsilon_r}\cdots L_{-\varepsilon_1}v_h + \sum_{\substack{0 \leqslant k < r \\ (i_1, \cdots, i_k) \in \bar{E}}} b_{\underline{i}}E_{-i_1}\cdots E_{-i_k}v_h$$

where $i_s \in I_+$, $\varepsilon_s \in M_+$ for any s and $\varepsilon_r < \cdots < \varepsilon_1$. Denote $\bar{I}_0 = \{(i_1, \cdots, i_k) \mid b_{\underline{j}} \neq 0\}$, $\underline{i}(0) = \min\{\varepsilon_r, i_1 \mid \underline{i} := (i_1, \cdots, i_k) \in \bar{I}_0\}$. Let $\varepsilon \in I_+$ such that $\varepsilon < \underline{i}(0)$. Considering the weight of u is λ, we have

$$E_{h-\lambda-\varepsilon}u = f(h-\lambda-\varepsilon)E_{-\varepsilon}v_h + \sum_{\substack{1 \leqslant k < r \\ (i_1, \cdots, i_k) \in \bar{E} \\ i_1, \cdots, i_k \in I_+}} b_{\underline{i}}g_{\underline{i}}(h-\lambda-\varepsilon)E_{-\varepsilon}v_h =$$

$$(f(h-\lambda-\varepsilon) + \sum b_{\underline{i}}g_{\underline{i}}(h-\lambda-\varepsilon))E_{-\varepsilon}v_h \in U(\mathrm{SVir}[M, \alpha])u_0$$

where $f(x)$, $g(x)$ are polynomials and $\deg f(x) = r$, $\deg g_{\underline{i}}(x) \leqslant r-1$. So we can find some ε such that $f(h-\lambda-\varepsilon) + \sum b_{\underline{i}}g_{\underline{i}}(h-\lambda-\varepsilon) \neq 0$. Thus $E_{-\varepsilon}v_h \in U(\mathrm{SVir}[M, \alpha])u_0$. The first part of claim 2 is proved.

Let $\varepsilon' \in I_+$ such that $\varepsilon' < \varepsilon$, then $E_{-\varepsilon'}v_h \in FE_{\varepsilon-\varepsilon'}E_{-\varepsilon}v_h \subset U(\mathrm{SVir}[M, \alpha])u_0$, so $E_{-(\varepsilon'+\varepsilon)}v_h \in F(E_{-\varepsilon}E_{-\varepsilon'}v_h - E_{-\varepsilon'}E_{-\varepsilon}v_h) \subset U(\mathrm{SVir}[M, \alpha])u_0$. Similarly, we can deduce that $E_{-x}v_h \in U(\mathrm{SVir}[M, \alpha])u_0$ for any $x \in B(\varepsilon)$. The second part of claim 2 is also proved.

For ε mentioned in claim 2 and for any $n \in N$, we have that ① If $n\varepsilon \in M_+$, then $E_{n\varepsilon}E_{-n\varepsilon}v_h = 2n\varepsilon\, L_0 v_h + \frac{n^3\varepsilon^3 - n\varepsilon}{12}\dot{c}v_h = \left(2n\varepsilon h + \frac{n^3\varepsilon^3 - n\varepsilon}{12}\dot{c}\right)v_h$; ② If $n\varepsilon \in M'_+$, then $G_{n\varepsilon}G_{-n\varepsilon}v_h = \left[2h - \frac{1}{3}\left((n\varepsilon)^2 - \frac{1}{4}\right)\dot{c}\right]v_h$. It is easy to see that, if $(\dot{c}, h) \neq (0, 0)$, there always exists some $n \in N$ such that $2nxh + \frac{n^3\varepsilon^3 - n\varepsilon}{12}\dot{c} \neq 0$ and $2h - \left(\frac{1}{3}(n\varepsilon)^2 - \frac{1}{4}\right)\dot{c} \neq 0$, so $v_h \in U(\mathrm{SVir}[M, \alpha])u_0$. Therefore, $M(\dot{c}, h)$ is irreducible.

Case 2 $\alpha \in M$.

Let $u_0 \neq 0$ be any weight vector.

Claim 1 There exists a weight vector $u \in U(\mathrm{SVir})u_0$ with weight λ such that for some $r \in N$

$$u \equiv (L_{-\varepsilon_r}L_{-\varepsilon_{r-1}}\cdots L_{-\varepsilon_1}v_h + \sum b_{\underline{\varepsilon}}E_{-\varepsilon_r}E_{-\varepsilon_{r-1}}\cdots E_{-\varepsilon_1}v_h)(\bmod V_{r-1}) \tag{6}$$

where the summands in the second summand of (6) are those of which at least one of $\{E_{-\varepsilon_j} \mid j = 1, \cdots, r\}$ has the form of $G_{-\varepsilon_j}$.

In fact, $u_0 \in V_r$ for some $r \in N$. Claim 1 holds clearly for $r = 0$ and $r = 1$. For $r > 1$, we set

$$u_0 \equiv \sum_{\substack{(i_1, \cdots, i_r) \in \bar{E} \\ i_1, \cdots, i_r \in I_+}} a_{\underline{i}}E_{-i_1}\cdots E_{-i_r}v_h(\bmod V_{r-1})$$

By a similar method as used in claim 1 and by picking $E_{j_r - \varepsilon_1}$ and $E_{j_{r-1} - \varepsilon_2}$ properly as in (4) and (5), we have

$$u'_1 = E_{j_r - \varepsilon_1}u_0 \equiv (\sum_{\substack{(\varepsilon_1, i_1, \cdots, i_{r-1}) \in \bar{E} \\ \varepsilon_1 \in M_+}} a^{(1)}_{\underline{i}}L_{-\varepsilon_1}E_{-i^{(1)}_1}\cdots E_{-i^{(1)}_{r-1}} + \sum b^{(1)}_{\underline{i}}G_{-\varepsilon_1}E'_{-i^{(1)}_1}\cdots E'_{-i^{(1)}_{r-1}})v_h(\bmod V_{r-1})$$

$$u'_2 = E_{j_{r-1} - \varepsilon_2}u'_1 \equiv \Big(\sum_{\substack{(\varepsilon_2, \varepsilon_1, i^{(2)}_1, \cdots, i^{(2)}_{r-1}) \in \bar{E} \\ \varepsilon_2, \varepsilon_1 \in M_+}} a^{(2)}_{\underline{i}}L_{-\varepsilon_2}L_{-\varepsilon_1}E_{-i^{(2)}_1}\cdots E_{-i^{(2)}_{r-1}} +$$

$$\sum b^{(2)}_{\underline{i}}E_{-\varepsilon_2}E_{-\varepsilon_1}E'_{-i_1(2)}\cdots E'_{-i_{r-1}(2)}\Big)v_h(\bmod V_{r-1}) \tag{7}$$

where the summands in the second summand of (7) are those of which at least one of $E_{-\varepsilon_1}$, $E_{-\varepsilon_2}$ has form of G_δ. We can prove by induction that

$$u' = u'_r \equiv (L_{-\varepsilon_r}\cdots L_{-\varepsilon_1} + \sum b_{\underline{i}}E_{-\varepsilon_r}\cdots E_{-\varepsilon_1})v_h(\bmod V_{r-1}) \tag{8}$$

Claim 2 There exists some $\varepsilon \in I_+$ such that

$$L_{-\varepsilon}v_h + dG_{-\varepsilon}v_h \in U(\mathrm{SVir}[M, \alpha])u_0$$

In fact, noting that the weight of u' is λ, we can choose $\varepsilon \in M_+$ such that

$$h - \lambda - \varepsilon \in I_+ \tag{9}$$

Let $L_{h-\lambda-\varepsilon}$ act on the two sides of (8), we have

$$L_{h-\lambda-\varepsilon}u' = f(h - \lambda - \varepsilon)L_{-\varepsilon}v_h + \sum g_{\underline{i}}(h - \lambda - \varepsilon)G_{-\varepsilon}v_h$$

where $f(h - \lambda - \varepsilon)$ and $g_{\underline{i}}(h - \lambda - \varepsilon) \in F$.

Since $\deg f(x) = r > 1$, $\deg g_{\underline{i}}(x) \leqslant r$, there exists $\varepsilon \in I_+$ such that

$$f(h - \lambda - \varepsilon) \neq 0 \tag{10}$$

Thus, $L_{-\varepsilon}v_h + \sum \frac{g_{\underline{i}}(h - \lambda - \varepsilon)}{f(h - \lambda - \varepsilon)}G_{-\varepsilon}v_h = L_{-\varepsilon}v_h + dG_{-\varepsilon}v_h \in U(\mathrm{SVir})u_0$. Therefore, if $d = 0$, by case 1, the sufficiency of the theorem holds. In the following, we suppose that $d \neq 0$.

By lemma 1, we have $L_\varepsilon L_{-\varepsilon}v_h + dL_\varepsilon G_{-\varepsilon}v_h = -2\varepsilon L_0 v_h + \frac{\varepsilon^3 - \varepsilon}{12}\dot{c}v_h + d\left(-\varepsilon - \frac{\varepsilon}{2}\right)G_0 v_h = \left(-2\varepsilon h + \frac{\varepsilon^3 - \varepsilon}{12}\dot{c}\right)v_h$. If $(h, \dot{c}) \neq (0, 0)$, since "$\succ$" is a dense order, there exists $\varepsilon \in I_+$ such that $-2\varepsilon h + \frac{\varepsilon^3 - \varepsilon}{12} \neq 0$ and (9) and (10) hold. So $v_h \in U(\mathrm{SVir})u_0$.

By case 1 and case 2, the sufficiency of theorem 3 is proved.

The necessity holds clearly. Otherwise, we can easily find a proper submodule of $M(\dot{c}, h)$, for example, $M'(0, 0) = \sum_{\substack{i_1, \cdots, i_k \in I_+ \\ k > 0}} FE_{-i_1}\cdots E_{-i_k}v_0$.

This completes the proof of theorem 3.

In a way similar to the ones used in Ref. [4] and in the proof of the above theorem, we can prove the following theorems. We only give the results, the details of the proofs are all omitted.

Theorem 4 If "$>$" is a dense order, then

$$M'(0,0)=\sum_{\substack{i_1,\cdots,i_k\in I_+\\k>0}} FE_{-i_1}\cdots E_{-i_k}v_0$$

is an irreducible submodule of $M(0,0)$ if and only if for any $x, y\in I_+$, there always exists a positive integer n such that $nx>y$.

Theorem 5 With respect to a discrete order "$>$", the Verma module $M(\dot{c}, h)$ is an irreducible SVir$[M, \alpha]$-module if and only if $M_a(\dot{c}, h)$ is an irreducible SVir$[2aZ, a]$-module.

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广义 Virasoro 超代数及其 Verma 模

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摘要: 对域 F 的加法子群 M 以及 $\alpha\in F$, 且 $2\alpha\in M$, 苏育才及赵开明定义了 2 类广义 Virasoro 超代数, 它们分别被记成 SVir$[M, \alpha]$和$\widehat{\text{SVir}}[M, \alpha]$, 后者是前者的平凡扩张. 基于对同构的讨论, 研究了 SVir$[M, \alpha]$的 Verma 模, 并且得到了这些模的不可约性.

关键词: 李代数; 模; 同构

中图分类号: O151.2; O152.5