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Three methods for generating monotonic OWA operator weights with given orness level

Liu Xinwang

(College of Economics and Management, Southeast University, Nanjing 210096, China)

Abstract: Based on the properties of ordered weighted averaging (OWA) operator and regular increasing monotone (RIM) quantifier, three methods for generating monotonic OWA operator weights are proposed. They are geometric OWA operator weights, equidifferent OWA operator weights and the modified RIM quantifier OWA weights. Compared with most of the common OWA methods for generating weights, the methods proposed in this paper are more intuitive and efficient in computation. And as there are more than one solution in most cases, the decision maker can set some initial condition and chooses the appropriate solution in the real decision process, which increases the flexibility of decision making to some extent. All these three OWA methods for generating weights are illustrated by numerical examples.

Key words: ordered weighted averaging operator; orness measure; fuzzy quantifier

The ordered weighted averaging (OWA) operators introduced by Yager[1] have attracted much interest among researchers. Yager[1] proposed orness measures of the weights vector. It is clear that the actual type of aggregation performed by an OWA operator depends upon the form of the weights vector[2,3]. A number of approaches have been suggested for obtaining the associated weights, i.e., quantifier guided aggregation[1,4,5], exponential smoothing[6], learning[7] and goal programming[8]. Hagan proposed a method to generate weights with a given orness level[9]. The problem is formulated as a nonlinear constraint program problem with maximum entropy procedure. The weights vector is called maximum entropy OWA (MEOWA) weights. Filev and Yager[6,10], Fullér and Majlender[11] further analyzed the properties of MEOWA operator and proposed the corresponding methods to get it. Recently, Liu and Chen extended the MEOWA operator to a more generic form and proposed an alternative method to get its weights[12].

In this paper, a class of OWA operators with monotonic weights is proposed, which means the weights change with the change of elements to be aggregated in the same or reverse order direction. The direct meaning of the monotonic OWA weights is that it reflects the decision maker's consistent preferences to the value of the aggregated elements: the bigger, the more important, or the smaller, the more important. Especially, two classes of monotonic OWA operators called equidifferent OWA operator and geometric OWA operator are researched. The former with weights composed of an arithmetic progression and 0s and the latter with weights with a fixed ratio. The methods for getting monotonic OWA weights with given orness degree with geometric and equidifferent OWA operator are proposed, respectively. In view of the close relationship between OWA operator and fuzzy quantifier, an intuition method of generating monotonic OWA weights with an RIM fuzzy quantifier is also proposed.

1 Some Properties of OWA Operator

An OWA operator[1] of dimension n is a mapping F: $\mathbf{R}^n \rightarrow \mathbf{R}$ that has an associated weighting vector $\boldsymbol{W} = \{w_1, w_2, \cdots, w_n\}$ with the following properties:

$$w_1 + w_2 + \cdots + w_n = 1,\ 0 \leqslant w_j \leqslant 1 \quad j = 1, 2, \cdots, n$$

and such that

$$F(x_1, x_2, \cdots, x_n) = \sum_{j=1}^{n} w_j y_j$$

with y_j being the j-th largest of x_i. Denote this expression as $F_W(X)$, where $X = \{x_1, x_2, \cdots, x_n\}$.

The degree of "orness" associated with this operator is defined as

$$\text{orness}(\boldsymbol{W}) = \sum_{j=1}^{n} \frac{n-j}{n-1} w_j \tag{1}$$

The min, max and average correspond to $\boldsymbol{W}^*$, $\boldsymbol{W}_*$ and $\boldsymbol{W}_A$, respectively, where $\boldsymbol{W}^* = \{1, 0, \cdots, 0\}^T$, $\boldsymbol{W}_* = \{0, 0, \cdots, 1\}^T$, $\boldsymbol{W}_A = \{1/n, 1/n, \cdots, 1/n\}^T$. Obviously, orness$(\boldsymbol{W}^*) = 1$, and orness$(\boldsymbol{W}_*) = 1$,

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Biography: Liu Xinwang (1968—), male, doctor, associate professor, xwliu@seu.edu.cn.

and orness$(\boldsymbol{W}_A) = 1/2$. The orness measure has the following property[5].

Proposition 1 Let $\boldsymbol{W} = \{w_1, w_2, \cdots, w_n\}$ be the weight vector of an OWA operator with orness$(\boldsymbol{W}) = \alpha$. Then for the reverse order of $\boldsymbol{W}$, $\boldsymbol{W}' = \{w_n, w_{n-1}, \cdots, w_1\}$, orness$(\boldsymbol{W}') = 1 - \alpha$.

In what follows, we will focus on a special class of OWA operators with monotonic weights. Some properties are listed. We will use these properties to get monotonic OWA weights. Their detailed explanations can be seen in Ref. [12].

Theorem 1[12] For OWA weights $\boldsymbol{W} = \{w_1, w_2, \cdots, w_n\}$, $\boldsymbol{W}' = \{w'_1, w'_2, \cdots, w'_n\}$, if $w_i/w_{i+1} \geqslant w'_i/w'_{i+1}$, with $i = 1, 2, \cdots, n-1$, then orness$(\boldsymbol{W}) \geqslant (>)$ orness$(\boldsymbol{W}')$ and for $\forall X = \{x_1, x_2, \cdots, x_n\}$, $F_{\boldsymbol{W}}(X) \geqslant F_{\boldsymbol{W}'}(X)$.

Theorem 2[12] For OWA weights $\boldsymbol{W} = \{w_1, w_2, \cdots, w_n\}$, $\boldsymbol{W}' = \{w'_1, w'_2, \cdots, w'_n\}$, if $w_i - w_{i+1} \geqslant w'_i - w'_{i+1}$, with $i = 1, 2, \cdots, n-1$, then orness$(\boldsymbol{W}) \geqslant$ orness$(\boldsymbol{W}')$ and for $\forall X = \{x_1, x_2, \cdots, x_n\}$, $F_{\boldsymbol{W}}(X) \geqslant F_{\boldsymbol{W}'}(X)$.

2 Generating of Geometric OWA Operator Weights

From theorem 1, in order to increase or decrease the orness degree of an OWA operator, the quotient of the adjacent weights should be increased or decreased. A geometric OWA operator is OWA operator that the weights have a geometric property with fixed ratio q for adjacent weights, that is $q = w_i / w_{i+1}$.

A geometric OWA operator can be expressed as

$$w_i = aq^{i-1} \qquad a > 0,\ q \geqslant 0 \tag{2}$$

Considering that $\sum_{n=1}^{n} w_i = 1$, then

$$w_i = \frac{q^{i-1}}{\sum_{j=0}^{n-1} q^j} \tag{3}$$

$$\text{orness}(\boldsymbol{W}) = \sum_{i=1}^{n-1} \frac{n-i}{n-1} w_i = \frac{\sum_{i=1}^{n-1}(n-i)q^{i-1}}{(n-1)\sum_{j=0}^{n-1} q^j} = \frac{\sum_{i=0}^{n-2}(n-1-i)q^i}{(n-1)\sum_{j=0}^{n-1} q^j} \tag{4}$$

With given n and orness$(\boldsymbol{W}) = \alpha$, q is the root of the solution of Eq.(5) which is transformed from Eq.(4).

$$(n-1)\alpha q^{n-1} - \sum_{i=2}^{n}((n-1)\alpha - i + 1)q^{n-i} = 0 \tag{5}$$

The process of generating geometric OWA weights $\boldsymbol{W}$ with a given orness degree α can be summarized as follows:

Algorithm 1

Case 1 If $\alpha = 0$ or $\alpha = 1$, set $\boldsymbol{W} = \boldsymbol{W}^*$ or $\boldsymbol{W} = \boldsymbol{W}_*$ directly.

Case 2 If $\alpha \in (0, 1)$, ① Determine q with Eq.(5); ② Determine w_i with Eq.(3).

Example 1 Generate MEOWA weights $\boldsymbol{W} = \{w_1, w_2, w_3\}$ with orness$(\boldsymbol{W}) = 1/3$. From Eq.(5), $2q^2 - q - 4 = 0$, $q = (\sqrt{33} + 1)/4$; from Eq.(3), it can be obtained

$$\boldsymbol{W} = \left\{\frac{1}{2} - \frac{\sqrt{33}}{18}, \frac{1}{3} + \frac{\sqrt{33}}{9}, \frac{5}{6} - \frac{\sqrt{33}}{18}\right\}$$

3 Generating of Equidifferent OWA Operator Weights

Similar to the geometric OWA operator, from theorem 2, in order to increase or decrease the orness degree of an OWA operator, we can also increase or decrease the difference of the adjacent weights. An equidifferent OWA operator is a class of monotonic OWA operator with weights that is composed of a series of nonnegative equidifferent real numbers and 0s. By setting the difference value between the adjacent weights, the OWA operator weights with different orness levels can be calculated.

Equidifferent OWA operator weights $\boldsymbol{W} = \{w_1, w_2, \cdots, w_n\}$ can be expressed as

$$w_i = \begin{cases} a + (i-1)d & a + (i-1)d \geqslant 0 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

with $\sum_{i=1}^{n} w_i = 1$.

If an equidifferent OWA operator has m nonnegative equidifferent elements, it can be expressed as the following two cases:

1) When $d \leqslant 0$,

$$w_i = \begin{cases} a + (i-1)d & 1 \leqslant i \leqslant m \\ 0 & m+1 \leqslant i \leqslant n \end{cases} \tag{7}$$

2) When $d \geqslant 0$,

$$w_i = \begin{cases} 0 & 1 \leqslant i \leqslant n-m \\ a - (n-i)d & n-m+1 \leqslant i \leqslant n \end{cases} \tag{8}$$

where $a > 0$.

It is obvious that the two forms of equidifferent OWA operator weights have reversed order. When the weighting vector has only one nonzero element, it can also be regarded as equidifferent combined with 0, we can always assume that $n \geqslant m \geqslant 2$.

In the following we will first consider the case of Eq.(7) with $d \leqslant 0$. To keep $\sum_{i=1}^{n} w_i = 1$, and $w_i \geqslant 0$ $(i = 1, 2, \cdots, n)$, from Eq.(7), a and d should satisfy the following conditions:

$$\sum_{i=1}^{m}(a+(i-1)d)=1 \tag{9}$$

$$a+(m-1)d\geqslant 0 \tag{10}$$

then

$$a=\frac{2+dm-dm^2}{2m} \tag{11}$$

$$\frac{2}{m-m^2}\leqslant d\leqslant 0 \tag{12}$$

With Eq.(11), orness($\boldsymbol{W}$) can be calculated as

$$\text{orness}(\boldsymbol{W})=\frac{12n-6-6m+dm-dm^3}{12(n-1)} \tag{13}$$

From Eq. (13), when n and m are specified, orness($\boldsymbol{W}$) is monotone decreasing with d, which means that when d changes from $\frac{2}{m-m^2}$ to 0, orness($\boldsymbol{W}$) changes from $\frac{3n-m-1}{3(n-1)}$ to $\frac{2n-m-1}{2(n-1)}$. Here another question arises, when m and d change, can orness($\boldsymbol{W}$) spread on $[1/2, 1)$? If it is true, considering Eq.(8) and proposition 1, for any given orness level $\Omega\in[0, 1]$, we can always find an equidifferent OWA operator $\boldsymbol{W}$, which makes orness($\boldsymbol{W}$) $=\Omega$. We will check it in the following.

For the interval series

$$D_m=\left[\frac{2n-m-1}{2(n-1)}, \frac{3n-m-1}{3(n-1)}\right] \qquad m=2, 3, \cdots, n$$

where $D_2=\left[\frac{2n-3}{2(n-1)}, 1\right]$, $\cdots$, $D_k=\left[\frac{2n-k-1}{2(n-1)}, \frac{3n-k-1}{3(n-1)}\right]$, $D_{k+1}=\left[\frac{2n-k-3}{2(n-1)}, \frac{3n-k-2}{3(n-1)}\right]$, $\cdots$, $D_n=\left[\frac{1}{2}, \frac{2n-1}{3(n-1)}\right]$, D_{k+1} is on the left side of D_k, and $\frac{3n-k-2}{3(n-1)}-\frac{2n-k-1}{2(n-1)}=\frac{k-1}{6(n-1)}>0$. They intersect each other, and cover the half unit interval $[1/2, 1]$. So for any $\Omega\in[1/2, 1]$, there will exist at least one equidifferent OWA operator $\boldsymbol{W}$, which makes orness($\boldsymbol{W}$) $=\Omega$, m can be determined with

$$\left.\begin{array}{l}\frac{2n-m-1}{2(n-1)}\leqslant\Omega\leqslant\frac{3n-m-1}{3(n-1)}\\ 2\leqslant m\leqslant n\end{array}\right\} \tag{14}$$

From Eq.(13), d can also be determined.

$$d=\frac{6(2n-m-1-2n\Omega+2\Omega)}{m(m^2-1)} \tag{15}$$

The process to generate equidifferent OWA operator weights $\boldsymbol{W}$ with a given orness level Ω can be summarized as follows:

Algorithm 2

Case 1 If $\Omega\in[1/2, 1]$, ① Determine m with Eq.(14); ② Determine d with Eq.(15); ③ Generate OWA weights from Eqs.(11) and (7).

Case 2 If $\Omega\in[0, 1/2]$, from proposition 1, we can firstly get equidifferent OWA operator weights $\hat{\boldsymbol{W}}$ with orness($\hat{\boldsymbol{W}}$) $=1-\Omega$, reverse the order of $\hat{\boldsymbol{W}}$, we can get $\boldsymbol{W}$ we wanted.

Example 2 Determine the equidifferent OWA operator weights $\boldsymbol{W}=\{w_1, w_2, \cdots, w_{10}\}$ with orness($\boldsymbol{W}$) $=7/9$.

From Eq.(14), we can get $5\leqslant m\leqslant 8$, m can be set as 5, 6, 7, 8.

1) When $m=5$, $d=0$, $a=1/5$,

$$\boldsymbol{W}=\left\{\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, 0, 0, 0, 0, 0\right\}$$

2) When $m=6$, $d=-1/35$, $a=5/21$,

$$\boldsymbol{W}=\left\{\frac{5}{21}, \frac{22}{105}, \frac{19}{105}, \frac{16}{105}, \frac{13}{105}, \frac{2}{21}, 0, 0, 0, 0\right\}$$

3) When $m=7$, $d=-1/28$, $a=1/4$,

$$\boldsymbol{W}=\left\{\frac{1}{4}, \frac{3}{14}, \frac{5}{28}, \frac{1}{7}, \frac{3}{28}, \frac{1}{14}, \frac{1}{28}, 0, 0, 0\right\}$$

4) When $m=8$, $d=-1/28$, $a=1/4$,

$$\boldsymbol{W}=\left\{\frac{1}{4}, \frac{3}{14}, \frac{5}{28}, \frac{1}{7}, \frac{3}{28}, \frac{1}{14}, \frac{1}{28}, 0, 0, 0\right\}$$

From the above discussion, it can be seen that for a given orness level α, there will often exist more than one m, which satisfy $\frac{2n-m-1}{2(n-1)}\leqslant\alpha\leqslant\frac{3n-m-1}{3(n-1)}$.

4 Generating of Operator Weight with Fuzzy Quantifier

The OWA operator weight vector was associated with fuzzy quantifier at the very beginning[1, 4, 5, 13 - 16]. Yager[4] proposed an OWA aggregation method with RIM quantifier and extended the orness measure of OWA operator to the RIM quantifier.

Given a linguistic quantifier Q, we can generate the OWA weights by $w_i=Q\left(\frac{i}{n}\right)-Q\left(\frac{i-1}{n}\right)$, then we can associate with this quantifier a degree of orness as orness(Q) $=\sum_{i=1}^{n}\frac{n-i}{n-1}\left(Q\left(\frac{i}{n}\right)-Q\left(\frac{i-1}{n}\right)\right)$, that is orness($Q$) $=\frac{1}{n-1}\sum_{i=1}^{n-1}Q\left(\frac{i}{n}\right)$. Let $n\to\infty$, then

$$\text{orness}(Q)=\int_0^1 Q(r)\,\mathrm{d}r \tag{16}$$

Thus the orness degree of an RIM linguistic quantifier is equal to the area under it.

In fact, when n is very large, in order to generate OWA weights with given orness level α, an RIM quantifier (x) with orness(Q) $=\alpha$ should be generated firstly, then the required OWA weights can be gotten from Q approximately. But in general, when n is not large enough, the generated OWA weights will not be the required even in an approximate way and we cannot guarantee that the weights are monotonic.

It is obvious that $\alpha = 0$, $\alpha = 1$, $\alpha = 1/2$ correspond to the unique OWA weight vector $\boldsymbol{W}_*$, $\boldsymbol{W}^*$ and $\boldsymbol{W}_A$, respectively. From proposition 1 we only need to consider that $\alpha \in (0, 1/2)$, and for $\alpha \in (1/2, 1)$ we only need to get the weights for $1-\alpha$ and reverse the order of them.

Here is an algorithm on how to generate the monotonic OWA weights with given orness degree $\alpha \in (1/2, 1)$. The main idea is to randomly generate monotonic increasing OWA weights $\boldsymbol{W} = \{w_1, w_2, \cdots, w_n\}$ $(w_1 \geqslant w_2 \geqslant \cdots \geqslant w_n)$. If its orness degree α' is greater than the given value α, decrease w_i with appropriate ratio α/α' except w_n; if α' is smaller than α, which means that the relative difference of w_i's is too big, add a positive real number to all w_i, then decrease w_i with an appropriate ratio to make $\sum_{i=1}^{n} w_i = 1$.

Algorithm 3

① Randomly generate $n+1$ nonnegative real number p_i, with increasing order that is $p_i - p_{i-1} > 0$, $(i = 1, 2, \cdots, n)$, and $p_0 = 0$;

② Calculate $q_i = \sum_{j=1}^{i} p_i$, $s_i = q_i/q_n$ and $\alpha' = \sum_{i=1}^{n-1} s_i/(n-1)$ $(i = 0, 1, \cdots, n)$, respectively;

③ If $\alpha' \geqslant \alpha$, goto ④, otherwise goto ⑥;

④ Calculate $s'_i = s_i\alpha (i = 0, 2, \cdots, n-1)$, where $\alpha = \alpha/\alpha'$, and $s'_n = s_n$;

⑤ Calculate $w_i = s'_i - s'_{i-1}$, end;

⑥ Let $s'_i = s_i + ir$, solve $\sum_{i=1}^{n-1} s'_i/s_n/(n-1) = \alpha$ for r;

⑦ Calculate $w_i = (s'_i - s'_{i-1})/s'_n$, end.

Example 3 Generate monotonic OWA weights $\boldsymbol{W} = \{w_1, w_2, w_3, w_4\}$ with orness$(\boldsymbol{W}) = 3/10$. The solution process is as follows:

① Randomly generate p_i which is increasing as $\boldsymbol{P} = \{0, 2, 3, 4, 5\}$;

② Calculate $\boldsymbol{Q} = \{0, 2, 5, 9, 14\}$ with $q_i = \sum_{j=0}^{i} p_i$, $i = 0, 1, 2, 3, 4$;

③ $s_i = q_i/q_4$ $(i = 0, 1, 2, 3, 4)$, $\boldsymbol{S} = \left\{0, \frac{1}{7}, \frac{5}{14}, \frac{9}{14}, 1\right\}$;

④ As $\alpha' = \frac{1}{3}(s_1 + s_2 + s_3) = \frac{1}{3} \times \left(\frac{1}{7} + \frac{5}{14} + \frac{9}{14}\right) = \frac{8}{21} > \frac{3}{10}$, $\alpha = \frac{3}{10}\Big/\frac{8}{21} = \frac{63}{80}$;

⑤ Calculate $s'_i = s_i\alpha$ $(i = 1, 2, 3)$ and $s'_4 = s_4$, get $\boldsymbol{S}' = \left\{0, \frac{9}{80}, \frac{9}{32}, \frac{81}{160}, 1\right\}$;

⑥ Calculate $w_i = s'_i - s'_{i-1}$, and get $\boldsymbol{W} = \left\{\frac{9}{80}, \frac{27}{160}, \frac{9}{40}, \frac{79}{160}\right\}$.

5 Conclusion

Based on the properties of OWA weights and RIM quantifier, this paper proposes three OWA methods for generating weights with the given orness level. They are geometric OWA operator weights, equidifferent OWA operator weights and the modified RIM quantifier OWA weights. Unlike most of the other OWA methods for generating weights which give a unique solution, the equidifferent OWA weights and quantifier methods for generating weights usually have more than one solution, which make the methods for generating weights more flexible, and allows the decision maker to select the solution he/she wants in real applications.

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给定 orness 水平下
生成单调 OWA 权值序列的 3 种方法

刘新旺

(东南大学经济管理学院, 南京 210096)

摘要: 基于 OWA 算子和 RIM 模糊量化算子的性质, 提出了在给定 orness 水平的情况下, 生成单调 OWA 算子权值序列的 3 种方法, 分别是等比 OWA 算子、等差 OWA 算子和基于修正的模糊量化算子的权值生成方法. 与现有的大多数方法相比, 该系列方法符合直觉且计算简单. 该系列方法还可以使决策者给出一定的初始条件, 而得到其所期望形式的 OWA 权值序列, 在一定程度上增加了决策的灵活性. 每种方法都给出了具体算例以表明其有效性和合理性.

关键词: OWA 算子; orness 测度; 模糊量化算子

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