Journal of Southeast University (English Edition) Vol. 20 No. 3 Sept. 2004 ISSN 1003—7985

A necessary and sufficient condition for a vertex-transitive graph to be star extremal

Lin Wensong Gu Guohua

(Department of Mathematics, Southeast University, Nanjing 210096, China)

Abstract: A graph is called star extremal if its fractional chromatic number is equal to its circular chromatic number. We first give a necessary and sufficient condition for a graph G to have circular chromatic number $|V(G)|/\alpha(G)$ (where $|V(G)|$ is the vertex number of G and $\alpha(G)$ is its independence number). From this result, we get a necessary and sufficient condition for a vertex-transitive graph to be star extremal as well as a necessary and sufficient condition for a circulant graph to be star extremal. Using these conditions, we obtain several classes of star extremal graphs.

Key words: circular chromatic number; fractional chromatic number; circulant graph; star extremal graph

Let k and d be natural numbers such that $k \geqslant 2d$. A (k, d)-coloring of a graph $G = (V, E)$ is a mapping $\varphi: V \rightarrow \{0, 1, 2, \cdots, k-1\}$ such that, for each edge $uv \in E$, $|\varphi(u) - \varphi(v)|_k \geqslant d$, where $|x|_k := \min\{|x|, k - |x|\}$. We say G is (k, d)-colorable if there exists a (k, d)-coloring of G. The circular chromatic number (also known as the star chromatic number which was first introduced by Vince[1]) $\chi_c(G)$ is the infimum of k/d for which G is (k, d)-colorable. For a different but equivalent definition of the circular chromatic number, we refer the reader to Ref. [2]. The circular chromatic number is a natural generalization of the ordinary chromatic number. Another generalization of the ordinary chromatic number is the fractional chromatic number of a graph. A mapping c from the collection Γ of independent sets of a graph G to the interval $[0, 1]$ is a fractional coloring if for every vertex x of G we have $\sum_{S \in \Gamma,\ \text{s.t.}\ x \in S} c(S) = 1$. The value of a fractional coloring c is $\sum_{S \in \Gamma} c(S)$. The fractional chromatic number $\chi_f(G)$ of G is the infimum of the values of fractional colorings of G. For equivalent definitions of the fractional chromatic number, see Refs. [3,4]. For any graph G, it is well known[5] that

$$\max\left\{\omega(G), \frac{|V(G)|}{\alpha(G)}\right\} \leqslant \chi_f(G) \leqslant \chi_c(G) \leqslant \lceil \chi_c(G) \rceil = \chi(G) \tag{1}$$

where $\alpha(G)$ is the independence number of G and $\omega(G)$ is the clique number of G.

Received 2004-04-05.

Foundation item: The National Natural Science Foundation of China (No. 10171013).

Biography: Lin Wensong (1968—), male, associate professor, wslin@seu.edu.cn.

A graph G is called star extremal if $\chi_c(G) = \chi_f(G)$. This notion of star extremality for graphs was first introduced by Gao and Zhu when they studied the chromatic numbers and the circular chromatic numbers of the lexicographic products of graphs in Ref. [6]. Klavžar[3] used star extremal graphs to investigate the chromatic numbers of lexicographic products of graphs. Lih, et al.[5] studied relations between circulant graphs and distance graphs and discussed their star extremality. Some other star extremal circulant graphs were also obtained by Lin[7].

Let p be a positive integer and let S be a subset of $\{1, 2, \cdots, p-1\}$. The circulant graph $G(p, S)$ is a graph with vertex set $V = \{0, 1, \cdots, p-1\}$ and edge set $E = \{ij: i, j \in V, |i-j|_p \in S\}$, where $|x|_p = \min\{|x|, p - |x|\}$. It is known and easy to prove that if a graph G is vertex-transitive, then $\chi_f(G) = |V(G)|/\alpha(G)$, where $\alpha(G)$ is the independence number of G. Since any circulant graph $G = G(p, S)$ is vertex-transitive, we have $\chi_f(G) = p/\alpha(G)$. Thus to prove that $\chi_c(G) = \chi_f(G)$ for a circulant graph $G = G(p, S)$, it is sufficient to prove that $\chi_c(G) = p/\alpha(G)$. Using the multiplier method, Refs. [5, 6] obtained some star extremal circulant graphs. The following two theorems will be used in our proofs.

Theorem 1[6] If $S = \{1, 2, \cdots, k-1\}$, then the circulant graph $G = G(p, S)$ is star extremal.

Theorem 2[6] Suppose that $k' = k + l \leqslant p/2$, $S = \{k, k+1, \cdots, k'\}$. If $p - 2k' < \min\{k, l\}$, then $G = G(p, S)$ is star extremal.

1 Graphs G with $\chi_c(G) = |V(G)|/\alpha(G)$

For two integers k and d with $k \geqslant 2d$, let G_k^d be the graph with vertex set $\{0, 1, 2, \cdots, k-1\}$ in which ij is an edge if and only if $d \leqslant |i-j| \leqslant k-d$.

Note that G_k^d is just the circulant graph $G(k, \{d, d+1, \cdots, \lfloor k/2 \rfloor\})$. Suppose that G and H are graphs. A homomorphism from G to H is a mapping f from $V(G)$ to $V(H)$ such that $f(x)f(y) \in E(H)$ whenever $xy \in E(G)$. If there is a homomorphism from G to H, we say G is homomorphic to H, denoted by $G \to H$. It is known and easy to prove that for any graph G, $\chi_c(G) \leqslant k/d$ if and only if G is homomorphic to G_k^d.

Let p and α be two positive integers with $p \geqslant 2\alpha$. Suppose $(p, \alpha) = \lambda$, let $p_0 = p/\lambda$ and $\alpha_0 = \alpha/\lambda$. And denote by S_i the set $\{(i-1)\lambda, (i-1)\lambda+1, \cdots, i\lambda-1\}$ $(i = 1, 2, \cdots, p_0)$. The graph $\widetilde{G_p^\alpha}$ has vertex set $\bigcup_{i=1}^{p_0} S_i$ and if $\alpha_0 \leqslant |i-j| \leqslant p_0 - \alpha_0$ then any vertex of S_i is adjacent to any vertex of S_j. Clearly there exists a homomorphism f: $\widetilde{G_p^\alpha} \to G_{p_0}^{\alpha_0}$ and $f(\widetilde{G_p^\alpha}) = G_{p_0}^{\alpha_0}$. When $(p, \alpha) = 1$, $\widetilde{G_p^\alpha}$ is exactly the graph G_p^α. If $\lambda \geqslant 2$, G_p^α is a proper subgraph of $\widetilde{G_p^\alpha}$. Note that the independence number of $\widetilde{G_p^\alpha}$ is α. And it is not difficult to check that $\widetilde{G_p^\alpha}$ is a vertex-transitive graph but not a circulant graph when $\lambda > 1$.

Theorem 3 Let G be a graph with p vertices and independence number α, then $\chi_c(G) = p/\alpha$ if and only if G is isomorphic to a spanning subgraph of $\widetilde{G_p^\alpha}$.

Proof Let $\lambda = (p, \alpha) \geqslant 1$. Let $p_0 = p/\lambda$ and $\alpha_0 = \alpha/\lambda$. If $\chi_c(G) = p/\alpha$, then there exists a homomorphism g: $G \to G_{p_0}^{\alpha_0}$. Denote by S_i the vertex set $\{v \mid g(v) = i, v \in V(G)\}$ $(i = 0, 1, 2, \cdots, p_0 - 1)$. Clearly $V(G) = \bigcup_{i=0}^{p_0-1} S_i$. Following we shall prove that $|S_i| = \lambda$ for $i = 0, 1, 2, \cdots, p_0 - 1$. Since $\bigcup_{j=i}^{i+\alpha_0-1} S_j$ is independent, we have $\sum_{j=i}^{i+\alpha_0-1} |S_j| \leqslant \alpha$ (where additions in indices are taken modulo p_0). Since $\sum_{i=0}^{p_0-1} \sum_{j=i}^{i+\alpha_0-1} |S_j| = \alpha_0 \sum_{j=0}^{p_0-1} |S_j| = \alpha_0 p = p_0 \alpha$, we have $\sum_{j=i}^{i+\alpha_0-1} |S_j| = \alpha$ $(i = 0, 1, 2, \cdots, p_0 - 1)$. Since $\sum_{j=0}^{\alpha_0-1} |S_j| = \sum_{j=1}^{\alpha_0} |S_j|$, we have $|S_0| = |S_{\alpha_0}|$. And since $\sum_{j=\alpha_0}^{2\alpha_0-1} |S_j| = \sum_{j=\alpha_0+1}^{2\alpha_0} |S_j|$, $|S_{\alpha_0}| = |S_{2\alpha_0}|$. Going in this way, it follows that $|S_0| = |S_{\alpha_0}| = |S_{2\alpha_0}| = |S_{3\alpha_0}| = \cdots = |S_{i\alpha_0}| = \cdots$. Note that $(p_0, \alpha_0) = 1$, we have $|S_0| = |S_1| = \cdots = |S_{p_0-1}| = \lambda$. And this implies that G is a spanning subgraph of $\widetilde{G_p^\alpha}$.

Conversely, if G is a spanning subgraph of $\widetilde{G_p^\alpha}$, then $\chi_c(G) \leqslant \chi_c(\widetilde{G_p^\alpha}) = p/\alpha$. Also, $p/\alpha \leqslant \chi_c(G)$ follows from (1).

2 Star Extremality of Graphs

In this section, we shall use theorem 3 to obtain some star extremal graphs which are not necessarily vertex-transitive. By (1), if $\chi_c(G) = |V(G)|/\alpha(G)$ then $\chi_c(G) = \chi_f(G)$ and G is star extremal. From theorem 3, we have the following theorem.

Theorem 4 Let G be a graph on p vertices with independence number α. If G is isomorphic to a spanning subgraph of $\widetilde{G_p^\alpha}$, then G is star extremal.

Since for any vertex-transitive graph G, we have $\chi_f(G) = |V(G)|/\alpha(G)$, thus by (1) the following theorem holds.

Theorem 5 Let G be a vertex-transitive graph (or especially a circulant graph) with vertex number p and independence number α. Then G is star extremal if and only if it is isomorphic to a spanning subgraph of $\widetilde{G_p^\alpha}$.

By the definitions of $\chi_c(G)$ and $\chi_f(G)$, if G is homomorphic to H, then $\chi_c(G) \leqslant \chi_c(H)$ and $\chi_f(G) \leqslant \chi_f(H)$. Using this simple observation and theorems 1 and 2, the following two theorems can be established. Let C_p^m denote the circulant graph $G(p, \{1, 2, \cdots, m-1\})$.

Theorem 6 Given two integers p and m with $p \geqslant 2m$. Let $p = \alpha m + q, 0 \leqslant q \leqslant m-1$. Then any graph G satisfying $C_p^m \to G \to \widetilde{G_p^\alpha}$ is star extremal.

Theorem 7 Suppose that $\alpha' = \alpha + l \leqslant p/2$ and $p - 2\alpha' < \min\{\alpha, l\}$, $S = \{\alpha, \alpha+1, \cdots, \alpha'\}$ and $G_1 = G(p, S)$. Then for any graph G satisfying $G_1 \to G \to \widetilde{G_p^\alpha}$, G is star extremal.

Proof From the proof of theorem 9 in Ref. [1], the independence number of G_1 is α and $\chi_c(G) = \chi_f(G) = p/\alpha$. And the theorem follows.

Corollary 1 Suppose that $\alpha' = \alpha + l \leqslant p/2$ and $p - 2\alpha' < \min\{\alpha, l\}$. Any circulant graph $G(p, S)$ with its generating set S satisfying

$$\{\alpha, \alpha+1, \cdots, \alpha'\} \subseteq S \subseteq \{\alpha, \alpha+1, \cdots, p-\alpha\}$$

is star extremal.

Corollary 2 Given two integers p and m with $p \geqslant 2m$. Let $p = \alpha m + q, 0 \leqslant q \leqslant m-1$. If $G(p, S)$ is a circulant graph and S satisfies

$$\{\alpha, 2\alpha, \cdots, (m-1)\alpha\} \subseteq S \subseteq$$

$\{\alpha, \alpha+1, \cdots, p-\alpha\}$

then G is star extremal.

Proof Let $S_1=\{\alpha, 2\alpha, \cdots, (m-1)\alpha\}$. We define a mapping from C_p^m to $G(p, S_1)$ as follows:

$f(i)=\alpha i \pmod p \qquad i=0, 1, 2, \cdots, p-1$

Since if $i-j\in\{1, 2, \cdots, m-1\}$ then $f(i)-f(j)\in S_1$, f is a homomorphism from C_p^m to $G(p, S_1)$. Thus if S satisfies the condition of the corollary, then $C_p^m\to G(p, S)\to G_p^\alpha$. By theorem 6, $G(p, S)$ is star extremal.

Corollary 3 Given two integers p and m. If $p=m\alpha+q$, $0\leqslant q\leqslant m-1$ and $(p, \alpha)=1$, then for any S satisfying

$$\{1, 2, \cdots, m-1\}\subseteq S\subseteq\{1, 2, \cdots, p-1\}\backslash\{i_1, i_2, \cdots, i_{\alpha-1}\}$$

the graph $G(p, S)$ is star extremal. Where i_j is the integer with $\alpha i_j \pmod p)=j$, $j=1, 2, \cdots, \alpha-1$.

Proof Let

$S_1=\{1, 2, \cdots, m-1\}$

$S_2=\{1, 2, \cdots, p-1\}\backslash\{i_1, i_2, \cdots, i_{\alpha-1}\}$

Define f: $G(p, S_2)\to G_p^\alpha$ as in the proof of corollary 2. Since $(p, \alpha)=1$, f is one to one and it is not difficult to check that f is an isomorphism from $G(p, S_2)$ to G_p^α. Thus if $S_1\subseteq S\subseteq S_2$, then $C_p^m\to G(p, S)\to G_p^\alpha$. From theorem 6, the corollary holds.

Theorem 8 Given two integers p and m with $p\geqslant 2m$ and $m\geqslant 2$. Suppose $p=\alpha m+q$, $0\leqslant q\leqslant m-1$. And let

$$S=\left\{1, 2, \cdots, m-1, \left\lceil\frac{p-l+2}{2}\right\rceil, \cdots, \left\lfloor\frac{p}{2}\right\rfloor\right\}$$

where $\left\lceil\frac{p-l+2}{2}\right\rceil\leqslant\left\lfloor\frac{p}{2}\right\rfloor$. If ① α is odd and $2\leqslant l\leqslant m$, or ② α is even and $q+2\leqslant l\leqslant m$, then $G(p, S)$ is star extremal.

Proof 1) α is odd and $2\leqslant l\leqslant m$

Clearly the independence number of G is at most α. On the other hand, it is not very difficult to check that the following vertex set A is an independent set of G with $|A|=\alpha$.

$$A=\left\{0, m, \cdots, \frac{\alpha-1}{2}m, 0+\bar{m}, m+\bar{m}, \cdots, \frac{\alpha-3}{2}m+\bar{m}\right\}$$

where $\bar{m}=p-\left\lceil\frac{p-m+2}{2}\right\rceil+1$. Thus the independence number of G is α and $\chi_{\mathrm{f}}(G)=p/\alpha$.

Now define f: $G\to G_p^\alpha$ as follows:

$f(i)=i\alpha \pmod p \qquad i=0, 1, 2, \cdots, p-1$

Let i and j be two arbitrary vertices of G and let $t=|i-j|_p$. If $ij\in E(G)$ (that is $t\in S$), we shall show that $f(i)f(j)\in E(G_p^\alpha)$. If $t\in\{1, 2, \cdots, m-1\}$, it is easy to see that $|f(i)-f(j)|_p\geqslant\alpha$. So we suppose $t\in\left\{\left\lceil\frac{p-l+2}{2}\right\rceil, \cdots, \left\lfloor\frac{p}{2}\right\rfloor\right\}$. It is clear that there exists some h $(2\leqslant h\leqslant l\leqslant m)$ such that $t=(p-h+2)/2$. Without loss of generality, assume $t=(p-h+2)/2$. Then

$$|f(i)-f(j)|_p=|t\alpha|_p=\left|\frac{p-h+2}{2}\alpha\right|_p=\left|\frac{p-(h-2)\alpha}{2}\right|_p\geqslant\alpha$$

Hence $f(i)f(j)\in E(G_p^\alpha)$ and this implies that f is a homomorphism from G to G_p^α. Thus $\chi_{\mathrm{c}}(G)\leqslant p/\alpha$. By (1), $\chi_{\mathrm{c}}(G)=\chi_{\mathrm{f}}(G)$ and G is star extremal.

2) α is even and $q+2\leqslant l\leqslant m$

We first show that $\alpha(G)=\alpha-1$. Let A be a maximum independent set of G. As G is vertex-transitive, we may assume $0\in A$. Let

$$I_1=\left\{0, 1, \cdots, \left\lceil\frac{p-l+2}{2}\right\rceil-1\right\}$$

and

$$I_2=\left\{p-\left\lceil\frac{p-l+2}{2}\right\rceil+1, p-\left\lceil\frac{p-l+2}{2}\right\rceil+2, \cdots, p-m\right\}$$

Since 0 is adjacent to every vertex of

$\{p-m+1, p-m+2, \cdots, p-1\}$

and

$$\left\{\left\lceil\frac{p-l+2}{2}\right\rceil, \left\lceil\frac{p-l+2}{2}\right\rceil+1, \cdots, p-\left\lceil\frac{p-l+2}{2}\right\rceil\right\}$$

It is clear that $A\subseteq I_1\cup I_2$. Note that every m consecutive vertices of G induces a clique, we have $|A\cap I_k|\leqslant\left\lceil\frac{|I_k|}{m}\right\rceil$, $k=1, 2$. But

$$\left\lceil\frac{|I_1|}{m}\right\rceil=\left\lceil\frac{\left\lceil\frac{p-l+2}{2}\right\rceil}{m}\right\rceil=\left\lceil\frac{\left\lceil\frac{m\alpha+q-l+2}{2}\right\rceil}{m}\right\rceil=\frac{\alpha}{2}+\left\lceil\frac{\left\lceil\frac{q-l+2}{2}\right\rceil}{m}\right\rceil=\frac{\alpha}{2}$$

and

$$\left\lceil\frac{|I_2|}{m}\right\rceil=\left\lceil\frac{\left\lceil\frac{p-l+2}{2}\right\rceil-m}{m}\right\rceil=\frac{\alpha}{2}-1$$

Hence $|A|\leqslant\alpha-1$. On the other hand, it is not difficult to check that the following vertex set A' of G is an independent set with $|A'|=\alpha-1$.

$$A'=\left\{0, m, \cdots, \left(\frac{\alpha}{2}-1\right)m, 0+\bar{l}, m+\bar{l}, \cdots, \left(\frac{p}{2}-2\right)m+\bar{l}\right\}$$

where $\bar{l}=p-\left\lceil\frac{p-l+2}{2}\right\rceil+1$. Thus $\alpha(G)=\alpha-1$.

And $\chi_f(G) = p/\alpha - 1$.

Define f: $G \to G_p^{\alpha-1}$ as follows:

$$f(i) = i(\alpha - 1)(\mathrm{mod}\ p) \qquad i = 0, 1, 2, \cdots, p-1$$

By very similar argument in the proof of 1), one can easily show that f is a homomorphism from G to $G_p^{\alpha-1}$. This implies that $\chi_c(G) \leqslant p/(\alpha - 1)$. By (1), $\chi_c(G) = \chi_f(G)$ and G is star extremal.

Remark 1 In the study of star extremal graphs, Klavžar[3] gave a family of circulant graphs $G_n = G(3n-1, \{1, 4, \cdots, 3n-2\})$ and proved that $\chi_c(G_n) = \chi_f(G_n) = 3 - 1/n$. However by defining a mapping f: $G_n \to G_{3n-1}^n$ such that for each vertex $i = 0, 1, 2, \cdots, 3n-2$, $f(i) = ni$ (mod $3n-1$), one can easily see that G_n is isomorphic to G_{3n-1}^n. While $\chi_c(G_k^d)$ was determined by Vince in 1988. Zhu[2] also mentioned this family of circulant graphs G_n viewing it as a new class of star extremal circulant graphs.

Remark 2 In some sense, theorems 4 and 5 determine all star extremal graphs which satisfy $|V(G)|/\alpha(G) = \chi_f(G) = \chi_c(G)$ and all star extremal vertex-transitive graphs, respectively. By (1), we ask how to determine all the star extremal graphs which satisfy $\chi_f(G) = \chi_c(G) = \chi(G)$.

References

[1] Vince A. Star chromatic number [J]. *J Graph Theory*, **1988, 12**(4): 551−559.

[2] Zhu X D. Circular chromatic number —a survey [J]. *Discrete Math*, **2001, 229**(1−3): 371−410.

[3] Klavžar S. A note on the fractional chromatic number and the lexicographic product of graphs [J]. *Discrete Math*, **1998, 185**(1−3): 259−263.

[4] Larsen M, Propp J, Ullman D. The fractional chromatic number of Mycielski's graphs [J]. *J Graph Theory*, **1995, 19**(4): 411−416.

[5] Lih K, Liu D D, Zhu X D. Star extremal circulant graphs [J]. *SIAM J Discrete Math*, **1999, 12**(4): 491−499.

[6] Gao G, Zhu X D. Star extremal graphs and the lexicographic product [J]. *Discrete Math*, **1996, 152**(1−3): 147−156.

[7] Lin W S. Some star extremal circulant graphs [J]. *Discrete Math*, **2003, 271**(1−3): 169−177.

顶点可迁图为 star extremal 的一个充要条件

林文松　　顾国华

(东南大学数学系, 南京 210096)

摘要: 一个图当它的圆色数和分色数相等称之为 star extremal. 本文首先给出一个图的圆色数等于顶点数除以独立数的充要条件. 然后利用这个结果给出了顶点可迁图是 star extremal 的一个充要条件. 并由此得到了几类新的 star extremal 图.

关键词: 圆色数; 分色数; 循环图; star extremal 图

中图分类号: O157. 5