

Ramsey numbers $r(K_{1,4}, G)$ for all three-partite graphs G of order six

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Abstract: In this paper, we use a combinatorial analysis method. In the complete graph K_N with edges colored arbitrarily by red or blue, we consider the proposition of the subgraph of the red graph or blue graph induced by the neighborhood of some vertex in $V(K_N)$. Inspired by the main results of Jayawardene and Rousseau (*Ars Combinatoria*, 2000, 163 – 173), we determine the Ramsey numbers of $r(K_{1,4}, G)$, where G is the three-partite graph of order six without isolate vertex.

Key words: Ramsey number; the graph of order six; three-partite graph

If the edges of the complete graph K_N are colored either red or blue, denote the spanning subgraph with all red edges and all blue edges, respectively, by R and B . Then R and B are called a factorization of K_N and denoted by $K_N = R \oplus B$.

For the graph G and H , the Ramsey number $r(G, H)$ is the least positive integer N such that if $K_N = R \oplus B$ is an arbitrary factorization, then either G is a subgraph of R or H is a subgraph of B .

All graphs in this paper are both finite and simple without isolate vertices. $E(G)$, $V(G)$ and $N_G(v)$ denote the edge set, vertex set and the neighborhood of vertex v in G . Maximum degree and minimum degree of a graph G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. Let B_n denote the book with n pages. Here $B_n = K_2 + \overline{K_n} = K_1 + K_{1,n} = K_{1,1,n}$. We refer reader to Ref. [1] for any notation and terminology not explained.

In Ref. [2], Jayawardene and Rousseau gave the Ramsey numbers of $r(C_5, G)$, when G was the graph of order six without isolate vertex. They specially proved that the situation G was the three-partite graph of order six without isolate vertex. Inspired by their results, we proved the similar results of other common five vertices graph $K_{1,4}$.

Lemma 1^[3] There are only two 3-regular graphs of order six (see Fig.1).

Since three-partite graph of order six without

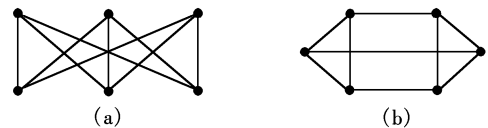


Fig.1 3-regular graphs of order six

isolate vertex can be only one of $K_{1,2,3}$, $K_{2,2,2}$ or B_4 , we mainly prove the situation of $K_{1,2,3}$ and $K_{2,2,2}$.

Theorem $r(K_{1,4}, G) = 11$, where G is the three-partite graph of order six.

We complete the proof of the theorem by the following two lemmas.

Lemma 2 $r(K_{1,4}, K_{1,2,3}) = r(K_{1,4}, K_{2,2,2}) = 11$.

Proof Considering a factorization $K_{10} = R \oplus B$, in which $R \cong H$ (H is isomorphic to Fig.2.), since $K_{1,4} \not\subseteq R$ and $K_{1,2,3}, K_{2,2,2} \not\subseteq B$, we have $r(K_{1,4}, K_{1,2,3}) = r(K_{1,4}, K_{2,2,2}) \geq 11$. Then we will prove the reverse inequalities respectively.

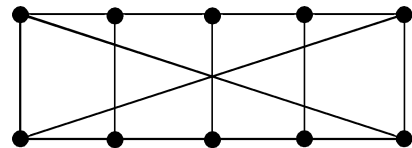


Fig.2 3-regular graph of order ten

Let $K_{11} = R \oplus B$ be an arbitrary factorization. If $K_{1,4} \not\subseteq R$, then $\Delta(R) \leq 3$.

Proposition^[3] Any undirected simple graph G has even vertices of odd degree.

By the proposition, the case that R has the most edges is only one vertex of degree two in R and the others of degree three. If in this case we have $K_{1,2,3} \subseteq B$ and $K_{2,2,2} \subseteq B$, then we also have them in the

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other cases.

Suppose $V(K_{11}) = \{v_1, v_2, \dots, v_{11}\}$. Let v_1 denote the unique vertex of degree two in R . Suppose v_2, v_3 are adjacent to v_1 in R , that is $N_R(v_1) = \{v_2, v_3\}$. Now there are two situations. One is that v_2 is adjacent to v_{10}, v_{11} in R , that is $N_R(v_2) = \{v_1, v_{10}, v_{11}\}$. The other is v_2 is adjacent to v_3, v_{11} in R , that is $N_R(v_2) = \{v_1, v_3, v_{11}\}$, and let v_{10} be another vertex adjacent to v_{11} in R . Because the proof of the second situation can be obtained from the similar proof of the first one and much simpler than the first one, we only prove the first situation here.

Now we have the new vertex set $V_M = N_B(v_1) \cap N_B(v_2) = \{v_4, v_5, v_6, v_7, v_8, v_9\}$. Let M denote the subgraph of R induced by V_M and \bar{M} denote its complement graph. The vertex set $V(K_{11})$ is labeled as Fig.3.

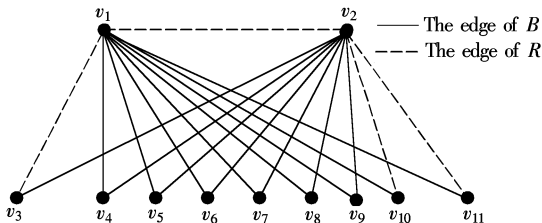


Fig.3 Graph of order eleven

Proof of $K_{1,2,3} \subseteq B$

Case 1 If $\{v_3, v_{10}, v_{11}\}$ induces a K_3 in R , then M is 3-regular of order six and isomorphic to Fig.1(a) or (b). V_M is labeled as Fig.4.

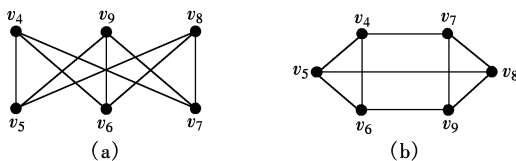


Fig.4 3-regular graphs of order six

Now we have the subgraph of B induced by $\{v_1, v_{10}, v_{11}, v_7, v_8, v_9\}$ contains $K_{1,2,3}$.

Case 2 If one of $\{v_3, v_{10}, v_{11}\}$, such as v_3 , is adjacent to one of V_M in R , without loss of generality, suppose $v_3v_4 \in E(R)$. Then the graph M in case 1 will lose one edge, and suppose that is v_4v_5 . Now $\{v_4, v_5, v_8, v_9\}$ spans $K_{1,3}$ in B . So the subgraph of B induced by $\{v_4, v_5, v_8, v_9, v_1, v_2\}$ contains $K_{1,2,3}$. To v_{10} and v_{11} , we have the same conclusion.

Excepting the former two cases, we all have $\bar{M} \supseteq K_{1,3}$, so the conclusion is obvious.

Proof of $K_{2,2,2} \subseteq B$

Case 1 If $\{v_3, v_{10}, v_{11}\}$ spans K_3 in R , M is 3-

regular of order six and isomorphic to Fig.1(a) or (b). V_M is labeled as Fig. 4 when M is isomorphic to Fig.1(a) or (b), the subgraph of B induced by $\{v_1, v_{10}, v_{11}, v_5, v_6, v_7\}$ contains $K_{2,2,2}$.

Case 2 If $\{v_3, v_{10}, v_{11}\}$ spans $K_3 - e$ in R , we consider the following two subcases.

(i) If $v_3v_{10} \in E(B)$, then v_3 and v_{10} are adjacent to V_M in R , respectively.

As M is isomorphic to the subgraph in Fig.4(a), if v_3 and v_{10} are exactly adjacent to two vertices of one edge of M in R , such as v_4 and v_5 , then M will only lose one edge compared with case 1. Now the subgraph of B induced by $\{v_1, v_3, v_5, v_{10}, v_6, v_7\}$ contains $K_{2,2,2}$. All the other cases that v_3 and v_{10} are adjacent to V_M in R will lead M to lose two edges compared with case 1. Then in those cases we all have $K_{2,2} \subseteq \bar{M}$, furthermore $K_{2,2,2} \subseteq B$.

As M is isomorphic to the subgraph in Fig.4(b), similarly, if v_3 and v_{10} are exactly adjacent to two vertices of one edge of M in R , such as v_4 and v_5 , the subgraph of B induced by $\{v_1, v_4, v_8, v_9, v_{10}, v_{11}\}$ contains $K_{2,2,2}$. The other cases are similar to the former proof.

When $v_3v_{11} \in E(B)$, we can prove in the same method.

(ii) If $v_{10}v_{11} \in E(B)$, then v_{10} and v_{11} are adjacent to V_M in R , respectively.

As M is isomorphic to the subgraph in Fig.4(a), if v_{10} and v_{11} are adjacent to two vertices of one edge of M in R , such as v_4 and v_5 , then M will only lose one edge compared with case 1. Now the subgraph of B induced by $\{v_1, v_{10}, v_5, v_{11}, v_6, v_7\}$ contains $K_{2,2,2}$. All the other cases that v_{10} and v_{11} are adjacent to V_M in R will lead M to lose two edges compared with case 1. Then in those cases we all have $K_{2,2} \subseteq \bar{M}$, furthermore $K_{2,2,2} \subseteq B$.

As M is isomorphic to the subgraph in Fig.4(b), similarly, if v_3 and v_{10} are exactly adjacent to two vertices of one edge of M in R , such as v_4 and v_5 , the subgraph of B induced by $\{v_1, v_9, v_4, v_{10}, v_5, v_{11}\}$ contains $K_{2,2,2}$. The other cases are similar to the former proof.

Case 3 If $\{v_3, v_{10}, v_{11}\}$ spans either $K_3 - 2e$ in R or K_3 in B , M will lose at least two edges compared with case 1. Now, either case will lead to $K_{2,2} \subseteq \bar{M}$, furthermore $K_{2,2,2} \subseteq B$.

By the proof of $K_{1,2,3} \subseteq B$ and $K_{2,2,2} \subseteq B$, we have $r(K_{1,4}, K_{1,2,3}) \leq 11$ and $r(K_{1,4}, K_{2,2,2}) \leq 11$.

Lemma 3 $r(K_{1,4}, B_4) = 11$.

In Ref. [4], Rousseau and Sheehan gave this result. In this paper, we give another way to prove it.

Proof Considering a factorization $K_{10} = R \oplus B$, in which $R \cong H$ (H is isomorphic to Fig.5), since $K_{1,4} \not\subseteq R$ and $B_4 \not\subseteq B$, $r(K_{1,4}, B_4) \geq 11$. Let $K_{11} = R \oplus B$ be an arbitrary factorization. If $K_{1,4} \not\subseteq R$, then $\Delta(R) \leq$

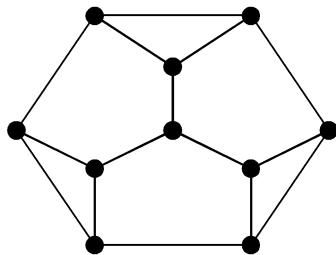


Fig.5 3-regular graph of order ten

3 and $\delta(B) \geq 7$. By $r(K_{1,4}, K_{1,4}) = 7^{[5]}$, for any vertex v of $V(K_{11})$, in its neighbor $N_B(v)$ we have $K_{1,4} \subseteq B$. Therefore, $K_1 + K_{1,4} = B_4 \subseteq B$.

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$K_{1,4}$ 和六阶三部图的 Ramsey 数 $r(K_{1,4}, G)$

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摘要: 本文主要应用了组合分析的方法. 在对所有边任意地进行了红、蓝 2 种颜色着色的完全图 K_N 中, 考察了完全图 K_N 的顶点集中某点的邻域在红图或蓝图中所生成的子图的性质. 在 Jayawardene 和 Rousseau (*Ars Combinatoria*, 2000, 163 – 173) 的主要结果的启发下, 研究并确定了另一种常见的五阶图 $K_{1,4}$ 对于所有无孤立点的六阶三部图 G 的 Ramsey 数 $r(K_{1,4}, G)$.

关键词: Ramsey 数; 六阶图; 三部图

中图分类号: O157.5