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Self-similar very singular solution of a p-Laplacian equation with gradient absorption: existence and uniqueness

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Abstract: This paper investigates the self-similar singular solution of the p-Laplacian evolution equation with the nonlinear gradient absorption terms $u_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u) - |\nabla u|^q$ for $1<p<2$ and $q>1$ in $\mathbf{R}^n \times (0, \infty)$. It has been proved that when $1<q<p-n/(n+1)$ there exists a unique self-similar very singular solution.

Key words: p-Laplacian evolution equation; gradient absorption; self-similar; singular solution; very singular solution

In this paper we study the self-similar singular solution of the p-Laplacian evolution equation with the nonlinear gradient absorption terms

$$u_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u) - |\nabla u|^q \qquad \text{in } \mathbf{R}^n \times (0, \infty) \tag{1}$$

where $1<p<2$ and $q>1$. In my previous paper[1] the case $p>2$ has been considered and the self-similar very singular solution has compact support.

Here by singular solution we mean a nonnegative and nontrivial solution $u(x, t)$ of Eq.(1) which is continuous in $\mathbf{R}^n \times [0, \infty) \setminus \{(0, 0)\}$ and satisfies

$$\lim_{t\to 0}\sup_{|x|>\varepsilon} u(x, t) = 0 \qquad \forall \varepsilon > 0 \tag{2}$$

A singular solution $u(x, t)$ is called a very singular solution provided that it satisfies

$$\lim_{t\to 0}\int_{|x|<\varepsilon} u(x, t)\,\mathrm{d}x = \infty \qquad \forall \varepsilon > 0 \tag{3}$$

By self-similar solution we mean that the solution $u(x, t)$ has the following special form:

$$u(x, t) = \left(\frac{\alpha}{t}\right)^{\alpha} v\left(|x|\left(\frac{\alpha}{t}\right)^{\alpha\beta}\right) \qquad \alpha = \frac{p-q}{2q-p},\ \beta = \frac{1+q-p}{p-q} \tag{4}$$

Here we restrict ourselves to the case $p>q$, which guarantees that α and β are positive. Thus, as a function of $r = |x|\left(\frac{\alpha}{t}\right)^{\alpha\beta}$, $v(\cdot)$ defined on $[0, \infty)$ solves

$$(|v'|^{p-2}v')' + \frac{n-1}{r}|v'|^{p-2}v' + \beta r v' + v - |v'|^q = 0 \qquad \forall r > 0 \tag{5}$$

It is easy to see that the condition (2) is equivalent to, if the solution u is given by Eq.(4),

$$\lim_{r\to\infty} r^{\frac{1}{\beta}} v(r) = 0 \tag{6}$$

and the condition (3) implies

$$\lim_{t\to 0} t^{n\beta-1}\int_{r<\varepsilon t^{-\beta}} v(r) r^{n-1}\,\mathrm{d}r = \infty \tag{7}$$

For $n\beta \geqslant 1$, Eq.(6) is not true. For $n\beta<1$, by Eq.(6) we see that $v \in L^1(0, \infty; r^{n-1}\mathrm{d}r)$ which shows that Eq.(7) holds automatically. That is, the singular solution is also self-similar very singular solution.

Singular solutions were first discovered for the semilinear heat equation $u_t = \Delta u - u^p$ in 1983 by Brezis and Friedman[2]. Since that time many authors studied the self-similar singular solutions of the following equations[3-7]:

$$u_t = \Delta(u^m) - u^q \qquad 0<m<\infty,\ q>1$$
$$u_t = \Delta(u^m) - |\nabla u|^p \qquad m \geqslant 1,\ p>1$$
$$u_t = \mathrm{div}(|\nabla u^m|^{p-2}\nabla u^m) - u^q \qquad 0<m<\infty,\ p>1,\ q>1$$

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For simplicity, throughout this paper we take

$$\sigma = 1 + q - p,\ \mu = p/(2-p),\ \theta = 1/(p-1) > 1,\ \eta = \mu - (\mu+1)q < 0$$
$$w^* = (\mu^{p-1}(\mu - n)/(\mu\beta - 1))^{1/(2-p)} \qquad \mu > n$$

To study the solution of Eq.(5) such that Eq.(6) is satisfied, we consider the initial value problem

$$\left.\begin{aligned} &(|v'|^{p-2}v')' + \frac{n-1}{r}|v'|^{p-2}v' + \beta r v' + v - |v'|^q = 0 \qquad r > 0 \\ &v(0) = a > 0,\ v'(0) = 0 \end{aligned}\right\} \tag{8}$$

1 Main Results

Theorem 1 Assume that $1 < q < p < 2$, β is given by Eq.(4) and $n\beta < 1$, i.e. $1 < q < p - n/(n+1)$. For each $a > 0$, let $v(r;\ a) = v(r)$ be the solution of Eq.(8). Then

1) There exists a unique finite positive number a^* such that $A = (0,\ a^*)$, $C = (a^*,\ \infty)$ and the following hold:

① If $a \in A$, then there is a finite number $R(a) > 0$ such that $v(R(a);\ a) = 0$, $v'(R(a);\ a) < 0$ and $v(r;\ a) > 0$, $v'(r;\ a) < 0$ for all $r \in (0,\ R(a))$.

② $v(r;\ a^*) > 0$, $v'(r;\ a^*) < 0$ for every $r \in (0,\ \infty)$. Moreover,

$$\lim_{r\to\infty} r^{\frac{1}{\beta}} v(r;\ a^*) = 0$$

that is, Eq.(1) has a unique self-similar very singular solution $u(x,\ t) = \left(\frac{\alpha}{t}\right)^{\alpha} v(r;\ a^*)$ with $r = |x|\left(\frac{\alpha}{t}\right)^{\alpha\beta}$.

③ If $a \in C$, then $v(r;\ a) > 0$, $v'(r;\ a) < 0$ for every $r \in (0,\ \infty)$. Moreover, there exists a constant $k(a) > 0$ such that

$$\lim_{r\to\infty} r^{\frac{1}{\beta}} v(r;\ a) = k(a)$$

2) As a function of a, $k(a)$ is continuous and strictly increasing such that

$$\lim_{a\to a^*} k(a) = 0,\ \lim_{a\to\infty} k(a) = \infty$$

2 Some Estimates of the Solution of (8)

Writing (8) as an equivalent integral equation, and using the standard Picard's iteration or the Banach fixed point theorem, we can prove that for each $a > 0$, (8) has a unique solution $v(r) = v(r;\ a)$, at least locally. Let $R(a) = \sup\{r > 0 \mid v(r) > 0\} > 0$, then $(0,\ R(a))$ is the maximal interval where $v > 0$. From (8), we see that $(|v'|^{p-2}v')'(0) = -\alpha a/n < 0$, and hence $v'(r) < 0$ in $(0,\ R(a))$. **It is easy to prove the following lemma:**

Lemma 1 1) Let v be the solution of (8) and $R(a) = \infty$. Then $\lim_{r\to\infty} v(r) = \lim_{r\to\infty} v'(r) = 0$.

2) Let $v(r;\ a)$ be the solution of (8). Then for every $r \in (0,\ R(a))$, $|v'(r)| \leqslant a^{1/q}$.

It follows from (8) that, near the origin,

$$v(r) = a - \Big[\frac{p-1}{p}\left(\frac{a}{n}\right)^{\theta} r^{p\theta} - \frac{p-1}{2p}\left(\frac{a}{n}\right)^{(3-p)\theta}\frac{\beta + (p-1)/p}{n(p-1)+p} r^{2p\theta} - \frac{p-1}{p+q}\left(\frac{a}{n}\right)^{(\sigma+1)\theta} \cdot \frac{1}{n(p-1)+q} r^{(p+q)\theta} + O(r^{(p+q)\theta+1})\Big] \tag{9}$$

By a variable transformation, we define

$$w(r) = w(r;\ a) = r^{\mu} v(r;\ a) \tag{10}$$

then a substitution of $v = r^{-\mu} w(r)$ into Eq.(5) yields

$$(p-1)r^2 w'' + [n - 1 - 2\mu(p-1)]rw' + \mu(\mu - n)w + |rw' - \mu w|^{2-p}[(1-\beta\mu)w + \beta r w' - r^{\eta}|rw' - \mu w|^q] = 0 \tag{11}$$

Note that $1 - \beta\mu = \dfrac{3p - 2p^2 - 2q}{(2-p)(p-q)} < 0$. Denote by $w_a = \dfrac{\partial w}{\partial a}$, then w_a satisfies the following linear equation:

$$\begin{aligned} L(w_a) := {} & (p-1)r^2 w_a'' + [n - 1 - 2\mu(p-1)]rw_a' + \mu(\mu-n)w_a + \\ & (2-p)|rw' - \mu w|^{-p}(rw' - \mu w)(rw_a' - \mu w_a)[(1-\beta\mu)w + \beta r w' - r^{\eta}|rw' - \mu w|^q] + \\ & |rw' - \mu w|^{2-p}[(1-\beta\mu)w_a + \beta r w_a' - q r^{\eta}|rw' - \mu w|^{q-2}(rw' - \mu w)\ (rw_a' - \mu w_a)] = 0 \end{aligned} \tag{12}$$

Lemma 2 If $w' > 0$ in a finite interval, $(0,\ r_1)$, $r_1 < R(a)$, then $\mu a w_a > r w'$ in $(0,\ r_1)$ and $w_a > 0$ on $[0,\ r_1]$.

Proof As $r(r^2 w'')' = r^2(rw')''$, using the differential operator $r\dfrac{d}{dr}$ to Eq.(11) yields

$$L(rw') = \eta r^{\eta} |rw' - \mu w|^{q+2-p} < 0 \qquad \text{in } (0,\ R(a))$$

In $(0,\ r_1)$, writing $w_a(r) = C(r)rw'(r)$ and applying the expansion (9), we can compute that

$$C(r) = \frac{v_a}{rv' + \mu v} = (\mu a)^{-1}\left[1 + \frac{q-1+(p-q)/\mu}{n(p-1)+q}\left(\frac{a}{n}\right)^{(\sigma+1)\theta} a^{-1} r^{(p+q)\theta} + O(r^{(p+q)\theta+1})\right]$$

near the origin. Thus $C(0) = (\mu a)^{-1}$ and $C'(r) > 0$ for all sufficiently small r. In addition, substituting $w_a = C(r)rw'$ into Eq. (12), we obtain that $C(r)$ satisfies

$$(p-1)(r^3 w')C''(r) + [\cdots]C'(r) + C(r)L(rw') = 0$$

Since $w' > 0$ and $L(rw') < 0$ in $(0,\ r_1)$, we see from the above that $C'(r)$ cannot have its first zero in $(0,\ r_1)$. Therefore, $C'(r) > 0$ in $(0,\ r_1)$ and $w_a = C(r)rw' > (\mu a)^{-1} rw'$ in $(0,\ r_1)$.

Now we prove that $w_a > 0$ at r_1. Let $r_0 = \min\{1,\ r_1/2\}$ and let ψ be the solution of the equation $L(\psi) = 0$ in $(0,\ R(a))$ with the initial data $\psi(r_0) = 0$ and $\psi'(r_0) = 1$. Then $\psi > 0$ in $(r_0,\ r_1]$ since the fact that between any two zeros of ψ there is a zero of w_a. Now we set $k_0 = C'(r)rw'(r)\,|_{r=r_0} > 0$ and $c_0 = C(r_0)$, then it follows that $L(\varphi) = 0$ in $(0,\ R(a))$ for $\varphi = w_a - k_0\psi$. Since $\varphi(r_0) = c_0\, rw'(r)\,|_{r=r_0}$, $\varphi'(r_0) = [C'(r)rw'(r) + C(r)(rw'(r))']\,|_{r=r_0} - k_0 = c_0(rw'(r))'\,|_{r=r_0}$. Therefore, we also write $\varphi = \tilde{C}(r)rw'(r)$, then $\tilde{C}(r)$ satisfies the same equation as C's, from which $\tilde{C}''(r_0) > 0$ since $\tilde{C}(r_0) = c_0 > 0$, and, $\tilde{C}'(r_0) = 0$. It follows that $\tilde{C}'(r) > 0$ in $(r_0,\ r_1)$ and $\varphi = \tilde{C}(r)rw' > 0$ in $[r_0,\ r_1)$. As a result, $w_a \geqslant k_0\psi > 0$ in $[r_0,\ r_1]$. This completes the proof of the lemma.

According to lemma 2, we can define three sets for every $a > 0$,

$$A = \{a > 0 \mid \text{There is an } R_1(a) \in (0,\ R(a)) \text{ such that } w'(R_1(a);\ a) = 0\}$$

$$B = \{a > 0 \mid w'(r;\ a) > 0 \text{ in } (0,\ \infty) \text{ and } \lim_{r\to\infty} w(r;\ a) < \infty\}$$

$$C = \{a > 0 \mid w'(r;\ a) > 0 \text{ in } (0,\ \infty) \text{ and } \lim_{r\to\infty} w(r;\ a) = \infty\}$$

Since $w'(r;\ a) > 0$ near the origin, if $a \notin A$ then $w' > 0$ in $(0,\ R(a))$, which implies that $R(a) = \infty$ and $a \in B \cup C$. Thus, A, B and C are disjoint to each other and $A \cup B \cup C = (0,\ \infty)$.

3 Properties of the Solution of (8) When $a \in A$

Lemma 3 Let $a > 0$, the following statements are equivalent:

① $a \in A$;

② There exists a finite $R_1 = R_1(a) \in (0,\ R(a))$ such that $w'(r;\ a) > 0$ in $(0,\ R_1)$, $w'(r;\ a) < 0$ in $(R_1,\ R(a))$ and $w''(R_1;\ a) < 0$;

③ $n\beta < 1$ and $\sup\limits_{r\in(0,\ R(a))} w(r;\ a) < w^*$;

④ $R(a) < \infty$ and $v'(R(a);\ a) < 0$.

Proof The proof is similar to that of lemma 3. 1 in Ref. [5], **we omit the details here.**

Theorem 2 Assume that $n\beta < 1$. Then there exists $a_* \in (0,\ \infty)$ such that $A = (0,\ a_*)$.

Proof $n\beta < 1$ implies $n < \mu$, i.e. $(n+1)p > 2n$. We show at first that A is a nonempty set and when $a \ll 1$, $(0,\ a) \subset A$. To do this, we consider the following initial data problem for sufficiently small ε:

$$\left.\begin{aligned} &(|v'|^{p-2}v')' + \frac{n-1}{r}|v'|^{p-2}v' + \beta rv + v - |v'|^q = 0 \\ &v(0) = \varepsilon,\ v'(0) = 0 \end{aligned}\right\} \tag{13}$$

Let $v(r;\ \varepsilon)$ be the solution of (13), $z_\varepsilon(t) = v(r;\ \varepsilon)/\varepsilon$, $t = r\varepsilon^{(2-p)/p}$. As a function of t, z_ε satisfies

$$\left.\begin{aligned} &(|z'_\varepsilon|^{p-2}z'_\varepsilon)' + \frac{n-1}{t}|z'_\varepsilon|^{p-2}z'_\varepsilon + \beta t z'_\varepsilon + z_\varepsilon - \varepsilon^{(2q-p)/p}|z'_\varepsilon|^q = 0 \\ &z_\varepsilon(0) = 1,\ z'_\varepsilon(0) = 0 \end{aligned}\right\} \tag{14}$$

Let $E_\varepsilon(t) = \frac{p-1}{p}|z'_\varepsilon|^p + \frac{1}{2}z_\varepsilon^2$, then by computing $E'_\varepsilon(t) < 0$ in $(0,\ R(a))$ which implies that $z_\varepsilon(t)$ and $z'_\varepsilon(t)$ are uniformly bounded with respect to $t \geqslant 0$ and $\varepsilon > 0$ and $|z'_\varepsilon| < \left(\frac{p}{2p-2}\right)^{1/p}$. Denote by $(0,\ T_\varepsilon)$ the maximal existence interval where $z_\varepsilon > 0$. Hence, $z'_\varepsilon < 0$ in $(0,\ T_\varepsilon)$. Now we consider

$$\left.\begin{aligned}&(|z'|^{p-2}z')'+\frac{n-1}{t}|z'|^{p-2}z'+\beta tz'+z=0\\&z(0)=1,\ z'(0)=0\end{aligned}\right\}\tag{15}$$

We claim that there is a finite $t_0>0$ such that the solution of (15) enjoys the property that $z(t_0)=0$, $z'(t_0)<0$ and $z(t)>0$, $z'(t)<0$ in $(0, t_0)$. In fact, by contradiction we can assume that $z(t)$ is positive in $(0, \infty)$. Then for every $t>0$,

$$|z'|^{p-2}v'(t)+\beta tz=\frac{1}{t^{n-1}}\int_0^t\rho^{n-1}(n\beta-1)z\mathrm{d}\rho<0\tag{16}$$

Take $t\geqslant 1$ such that $\int_0^t\rho^{n-1}(n\beta-1)z<-\delta$ for some constant $\delta>0$, that is, we have

$$|z'|^{p-2}v'(t)+\beta tz<-\delta t^{1-n}\qquad t\geqslant 1\tag{17}$$

For $n=1$, from (17) we have $z'(t)<-\delta^{\theta}$, and $z(t)\leqslant 0$ for all suitably large t. For $n\geqslant 2$, since $p/n\in(0, 1)$, applying the inequality $\beta tz+\delta t^{1-n}>(\beta tz)^{1-p/n}(\delta t)^{p/n}=\beta^{1-p/n}\delta^{p/n}t^{1-p}z^{1-p/n}$ yields

$$-z'>(\beta^{1-p/n}\delta^{p/n})^{\theta}t^{-1}z^{1-\nu}\qquad t\geqslant 1$$

for $\nu=\dfrac{(n+1)p-2n}{n(p-1)}>0$. From which we see that $v^{\nu}(t)<v^{\nu}(1)-\nu(\beta^{1-p/n}\delta^{p/n})^{\theta}\log t<0$ for all suitably large t. In each case, it contradicts the previous assumption that z is positive. Moreover, by sending $t\to t_0$ in (16) we deduce $z'(t_0)<0$. Therefore, the assertion holds.

Now we choose $0<\varepsilon,\ \zeta_0\ll 1$ satisfying

$$\begin{cases}0<t_0-t_1\ll 1,\ \varepsilon\leqslant\left(\dfrac{t_1}{t_2}\right)^{(n-1)p/(2q-p)} & \text{for } t_2\gg 1\\ z(t_1)<\zeta_0,\ z'(t_1)<\dfrac{1}{2}z'(t_0) & \\ \dfrac{1}{2^{p-1}}|z'|^{p-2}z'(t_0)+\beta t_0\zeta_0+\left(\dfrac{p}{2p-2}\right)^{(q-1)/p}\zeta_0<0 & \end{cases}$$

By the continuous dependence of the solution on the parameter ε, $T_{\varepsilon}>t_1$ and $z_{\varepsilon}(t_1)=\zeta<\zeta_0$, $z'_{\varepsilon}(t_1)<z'(t_0)/2$. Thus, since $|z'_{\varepsilon}|<\left(\dfrac{p}{2p-2}\right)^{1/p}$, from (14) we have

$$(t^{n-1}|z'_{\varepsilon}|^{p-2}z'_{\varepsilon}+\beta t^nz_{\varepsilon})'<t^{n-1}\left[(n\beta-1)z_{\varepsilon}-\varepsilon^{(2q-p)/p}\left(\frac{p}{2p-2}\right)^{(q-1)/p}z'_{\varepsilon}\right]$$

Integrating the above from t_1 to $t<\min\{T_{\varepsilon},\ t_2\}$ yields

$$\begin{aligned}t^{n-1}|z'_{\varepsilon}|^{p-2}z'_{\varepsilon}(t)+\beta t^nz_{\varepsilon}(t)&<t_1^{n-1}|z'_{\varepsilon}|^{p-2}z'_{\varepsilon}(t_1)+\beta t_1^nz_{\varepsilon}(t_1)+\varepsilon^{(2q-p)/p}\left(\frac{p}{2p-2}\right)^{(q-1)/p}t^{n-1}z_{\varepsilon}(t_1)<\\&t_1^{n-1}\left[2^{1-p}|z'|^{p-2}z'(t_0)+\beta t_0\zeta_0+\left(\frac{p}{2p-2}\right)^{(q-1)/p}\zeta_0\right]:=-\delta<0\end{aligned}$$

Hence, $|z'_{\varepsilon}|^{p-2}z'_{\varepsilon}+\beta tz_{\varepsilon}<-\delta t^{1-n}$ for all $t\in(t_1, \min\{T_{\varepsilon}, t_2\})$. By the same argument as in (17), if $T_{\varepsilon}=\infty$, then there is $\hat{t}=\hat{t}(\delta, \beta, n, p)$ such that $z_{\varepsilon}(\hat{t})=0$ provide that t_2 is suitably large, where ε is sufficiently small so that $\hat{t}\in(t_1, t_2)$. This is a contradiction. Therefore, $T_{\varepsilon}<\infty$ and $z'_{\varepsilon}(T_{\varepsilon})<0$. Thus for the initial value problem (13), by lemma 3, $(0, \varepsilon)\subset A$ when $\varepsilon\ll 1$ and A is nonempty.

Secondly, we need to show A is an open interval. If $\tilde{a}\in A$, then $w'(R_1(\tilde{a});\tilde{a})=0$ and $w''(R_1(\tilde{a});\tilde{a})<0$. By the implicit function theorem, the equation $w'(R_1;a)=0$ has a local unique C^1 solution $R_1=R_1(a)$ in a neighborhood of $\tilde{a}$ such that $w'(R_1(a);a)=0$ in such neighborhood, which implies that A is an open set and $R_1(a)$ is a C^1-function in A. Moreover, writing $m(a)=w(R_1(a);a)$ for $a\in A$, it follows that $\dfrac{\mathrm{d}m(a)}{\mathrm{d}a}=w'(R_1;a)\dfrac{\mathrm{d}R_1(a)}{\mathrm{d}a}+w_a=w_a>0$. Thus, if $(a_1, a_2)\subset A$ and $a_1>0$ then $a_1\in A$. In fact, by the continuous dependence of the solution on the initial data,

$$\sup_{r\in(0,R(a_1))}w(r;a_1)\leqslant\lim_{a\searrow a_1}m(a)<m((a_1+a_2)/2)<w^*$$

so that by lemma 3, $a_1\in A$ and A is an open interval. This completes the proof.

4 Properties of the Solution of (8) When $a \in B \cup C$

Lemma 4 Let $a > 0$. Then $a \in C \Leftrightarrow \sup\limits_{r \in (0, R(a))} w(r; a) > w^*$.

Proof The "$\Rightarrow$" part follows directly from the definition of set C. If $\sup\limits_{r \in (0, R(a))} w(r; a) > w^*$ holds, then by lemma 3, $a \notin A$ so that $a \in B \cup C$ and $w(r; a)$ is strictly increasing in $(0, \infty)$. If $\hat{w} = \lim\limits_{r\to\infty} w$ is finite, proceeding the proof of lemma 3.1 in Ref. [5], there is a sequence $\{r_j\}$ such that $((rw')\,|_{r=r_j}, (r^2 w'')\,|_{r=r_j}) \to (0, 0)$. It follows from Eq.(11) that $\hat{w} = w^*$ by sending $j \to \infty$, which contradicts the assumption. Hence, $\lim\limits_{r\to\infty} w = \infty$ and $a \in C$.

Theorem 3 There exists $a^* > 0$ such that $C = (a^*, \infty)$. Moreover, for every $a \in C$ there is some constant $k(a) > 0$ such that

$$\lim_{r\to\infty} r^{\frac{1}{\beta}} v(r; a) = k(a) \tag{18}$$

In addition, as a function of a, $k(a)$ is continuous and strictly increasing and satisfies

$$\lim_{a \searrow a^*} k(a) = 0, \quad \lim_{a \nearrow \infty} k(a) = \infty \tag{19}$$

Proof The proof is similar to that of theorem 4.1 in Ref. [5], we omit the details.

By theorems 2 and 3, $B = [a_*, a^*]$. Moreover, we have $a \in B \Leftrightarrow \sup\limits_{r \in (0, R(a))} w(r; a) = w^*$. By the definition of B, $\lim\limits_{r\to\infty} w(r; a) = w^*$ for $a \in B$.

Theorem 4 Let $n\beta < 1$. Then $a_* = a^*$, that is, $B = \{a_*\} = \{a^*\}$ and $\lim\limits_{r\to\infty} r^{1/\beta} v(r) = 0$.

Proof Since $\mu > 1/\beta$, We only need to prove $a_* = a^*$. We first show that $\lim\limits_{r\to\infty} rw'(r; a) = 0$ for $a \in B$. We consider a function defined in $[0, \mu)$,

$$f(\rho) = (n - \mu p)\rho + |\rho - \mu|^{2-p} w^{*2-p} [1 - \beta(\mu - \rho)] + \mu(\mu - n)$$

Then $f(0) = 0$ and for every $\rho \in (0, \mu)$,

$$f'(\rho) = (n - \mu p) + \beta(3-p)(\mu - \rho)^{2-p} w^{*2-p} - (2-p)(\mu-\rho)^{1-p} w^{*2-p} < (n - \mu p) + \beta(3-p)(\mu w^*)^{2-p} - (2-p)\mu^{1-p} w^{*2-p} = n - \mu p + (\beta\mu - 1)(2-p)\mu^{1-p} w^{*2-p} + \beta(\mu w^*)^{2-p} = f'(0)$$

There are two cases: ① $f'(0) > 0$ and ② $f'(0) \leqslant 0$. For case ①, using the arguments in the proof of theorem 3, writing $\tau = \ln r$ and $v(e^\tau) = v(1)\exp\left[-\int_0^\tau \Lambda(s)\,ds\right]$, then Λ satisfies $0 < \Lambda < \mu$ and

$$(p-1)\dot{\Lambda} = J(\Lambda, \tau) := (p-1)\Lambda^2 + (p-n)\Lambda + \Lambda^{2-p}[1 - \beta\Lambda - \Lambda^q e^{-q\tau} v^{q-1}(e^\tau)] w^{2-p}(e^\tau) \tag{20}$$

For every $0 < \varepsilon < \mu - n$ sufficiently small, $f(\varepsilon) > 0$. Then $\overline{\Lambda}(\tau) = \mu - \varepsilon$ is a sub-solution of Eq.(20). In fact, choosing T sufficiently large yields, for $\tau > T$,

$$(p-1)\frac{d\overline{\Lambda}(\tau)}{d\tau} - J(\overline{\Lambda}, \tau) < -f(\varepsilon) < 0$$

In addition, $w \to w^*$ indicates that there is at least a sequence $\{\tau_j\}$ such that $\Lambda(\tau_j) \to \mu$. By comparison, $\mu - \varepsilon \leqslant \Lambda < \mu$ for any small ε and sufficiently large τ, which implies that $\Lambda(\tau) \to \mu$ as $\tau \to \infty$ so that $rw' = (\mu - \Lambda)w \to 0$ as $r \to \infty$. For case ②, the equation $f(\rho) = 0$ has a unique root $\rho_0 = 0$ in $[0, \mu]$. On the other hand, there is $\{r_j \to \infty\}$ such that $\lim\limits_{j\to\infty} r_j w'(r_j) = b_0 \in [0, \mu w^*]$ exists, moreover, $w^* f(b_0/w^*) = 0$. Hence, $b_0 = 0$, i.e., $\lim\limits_{r\to\infty} rw' = 0$. Secondly, denote $\tau = \ln r$ and $rw' = \dot{w}$, when τ is sufficiently large, the linear operator L in Eq.(12) becomes

$$L(\phi) = (p-1)\ddot{\phi} + [b + o(1)]\dot{\phi} - [c + o(1)]\phi$$

where b is a certain constant, $c = p(\mu - n) > 0$ and $o(1) \to 0$ as $\tau \to \infty$. It follows that the solution of $L(\phi_1) = 0$ in (T, ∞) with the initial data $\phi_1(T) = 0$, $\dot{\phi}(T) = 1$ with T sufficiently large has the property that $\phi_1 \to \infty$ at exponentially rate as $\tau \to \infty$. On the other hand, the function ψ constructed in the proof of lemma 2 is positive in (r_0, ∞). Since w_a and ψ are linearly independent, one of them must be unbounded. Note that $w_a > k_0\psi$, we see that $w_a \to \infty$ as $r \to \infty$. As a result, if $a_* < a^*$, then by Fatou's lemma,

$$0 = \lim_{r\to\infty}[w(r; a^*) - w(r; a_*)] = \lim_{r\to\infty}\int_{a_*}^{a^*} w_a(r; a)\,da \geqslant \int_{a_*}^{a^*} \liminf_{r\to\infty} w_a(r; a)\,da = \infty$$

It is a contradiction. Thus, $a^* = a_*$. This completes the proof of the theorem.

Proof of Theorem 1 The conclusions of theorem 1 follow directly from theorems 2 to 4.

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p-拉普拉斯方程的自相似奇性解: 存在惟一性

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摘要: 研究了 p-拉普拉斯发展方程 $u_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u) - |\nabla u|^q$ 的自相似奇性解, 其中 $1 < p < 2$, $q > 1$, $(x, t) \in \mathbf{R}^n \times (0, \infty)$. 证明了当 $1 < q < p - n/(n+1)$ 时方程存在惟一的自相似强奇性解.

关键词: p-拉普拉斯发展方程; 梯度吸收; 自相似; 奇性解; 强奇性解

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