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On the uniqueness of entire functions

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Abstract: We study the uniqueness of entire functions and prove the following theorem: Let $f(z)$ and $g(z)$ be two nonconstant entire functions; n and k two positive integers with $n>2k+4$. If the zeros of both $f(z)$ and $g(z)$ are of multiplicity at least n, and $f^{(k)}(z)$ and $g^{(k)}(z)$ share 1 CM, then either $f(z)=c_1\mathrm{e}^{cz}$, $g(z)=c_2\mathrm{e}^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k c_1c_2c^{2k}=1$, or $f(z)\equiv g(z)$.

Key words: entire function; sharing value; uniqueness

Let f be a nonconstant meromorphic function in the whole complex plane. We shall use the following standard notations of value distribution theory[1,2]:

$$T(r,f),\ m(r,f),\ N(r,f),\ \overline{N}(r,f),\ \cdots$$

We denote by $S(r,f)$ any function satisfying

$$S(r,f)=o(T(r,f))$$

as $r\to\infty$, possibly outside of a set with finite measure.

Let a be a finite complex number, and k be a positive integer. We denote by $N_{k)}(r,1/(f-a))$ the counting function for zeros of $f(z)-a$ with multiplicity $\leqslant k$, and by $\overline{N}_{k)}(r,1/(f-a))$ the corresponding one for which multiplicity is not counted. Let $N_{(k}(r,1/(f-a))$ be the counting function for zeros of $f(z)-a$ with multiplicity $\geqslant k$, and $\overline{N}_{(k}(r,1/(f-a))$ the corresponding one for which multiplicity is not counted. Set $N_k(r,1/(f-a))=\overline{N}(r,1/(f-a))+\overline{N}_{(2}(r,1/(f-a))+\cdots+\overline{N}_{(k}(r,1/(f-a))$.

Let k be a positive integer. Set

$$E_k(a,f)=\{z\mid f(z)-a=0,\ \exists i,\ 1\leqslant i\leqslant k,\ \text{s.t.}\ f^{(i)}(z)\neq 0\}$$

where a zero point with multiplicity $m(\leqslant k)$ is counted m times in the set.

Let g be a nonconstant meromorphic function and a be a complex number. If $f(z)-a$ and $g(z)-a$ assume the same zeros with the same multiplicities, then we say that f and g share the value a CM. If $f(z)-a$ and $g(z)-a$ assume the same zeros ignoring the multiplicity, then we say that f and g share the value a IM.

Let f and g share 1 IM. We denote by $\overline{N}_{\mathrm{L}}(r,1/(f-1))$ the counting function for common 1-points of both f and g about which f has larger multiplicity than g, with multiplicity not being counted, and denote by $N_{11}(r,1/(f-1))$ the counting function for common simple 1-points of f and g. Similarly we have the notation $\overline{N}_{\mathrm{L}}(r,1/(g-1))$. Especially, if f and g share 1 CM, then $\overline{N}_{\mathrm{L}}(r,1/(f-1))=\overline{N}_{\mathrm{L}}(r,1/(g-1))=0$.

Hayman[3] and Clunie[4] proved the following result:

Theorem 1 Let f be a transcendental entire function, $n\geqslant 1$ a positive integer. Then $f^nf'=1$ has infinitely many solutions.

Fang and Hua[5], Yang and Hua[6] obtained a unicity theorem corresponding to the above result.

Theorem 2 Let f and g be two nonconstant entire functions, $n\geqslant 6$ a positive integer. If f^nf' and g^ng' share 1 CM, then either $f(z)=c_1\mathrm{e}^{cz}$, $g(z)=c_2\mathrm{e}^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1c_2)^{n+1}c^2=-1$, or $f\equiv tg$ for a constant t such that $t^{n+1}=1$.

Chen[7] and Wang[8,9] extended theorem 1 by proving the following theorem:

Theorem 3 Let f be a transcendental entire function, and n and k two positive integers with $n\geqslant k+1$. If the zeros of f are of multiplicity at least n, then $f^{(k)}=1$ has infinitely many solutions.

Naturally, we ask by theorem 1 and theorem 2 whether there exists a corresponding unicity theorem to theorem 3. In this paper, we give a positive answer to the above question by proving the following theorem.

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Theorem 4 Let f and g be two nonconstant entire functions, and let n and k be two positive integers with $n>2k+4$. If the zeros of both f and g are of multiplicity at least n, and $f^{(k)}$ and $g^{(k)}$ share 1 CM, then either $f(z)=c_1e^{cz}$, $g(z)=c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k c_1c_2c^{2k}=1$, or $f\equiv g$.

In fact, we prove the following more general results.

Theorem 5 Let f and g be two nonconstant entire functions, and let n and k be two positive integers with $n>4k+6$. If the zeros of both f and g are of multiplicity at least n, and $E_1(1, f^{(k)})=E_1(1, g^{(k)})$, then either $f(z)=c_1e^{cz}$, $g(z)=c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k c_1c_2c^{2k}=1$, or $f\equiv g$.

Theorem 6 Let f and g be two nonconstant entire functions, and let n and k be two positive integers with $n>(5k+9)/2$. If the zeros of both f and g are of multiplicity at least n, and $E_2(1, f^{(k)})=E_2(1, g^{(k)})$, then either $f(z)=c_1e^{cz}$, $g(z)=c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k c_1c_2c^{2k}=1$, or $f\equiv g$.

Theorem 7 Let f and g be two nonconstant entire functions, and let m, n and k be three positive integers with $m\geqslant 3$, $n>2k+4$. If the zeros of both f and g are of multiplicity at least n, and $E_m(1, f^{(k)})=E_m(1, g^{(k)})$, then either $f(z)=c_1e^{cz}$, $g(z)=c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k c_1c_2c^{2k}=1$, or $f\equiv g$.

From theorem 7, we get theorem 4, and for $k=1$, we get the results of Zhang[10] from theorems 5 to 7.

Theorem 8 Let f and g be two nonconstant entire functions, and let n and k be two positive integers with $n>5k+7$. If the zeros of both f and g are of multiplicity at least n, and $f^{(k)}$ and $g^{(k)}$ share 1 IM, then either $f(z)=c_1e^{cz}$, $g(z)=c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k c_1c_2c^{2k}=1$, or $f\equiv g$.

Let $k=1$. Then by theorem 8 we get the result of Xu and Qiu[11].

In order to prove our results, we firstly prove the following proposition:

Proposition 1 Let f and g be two nonconstant entire functions, and let k be a positive integer. If the zeros of both f and g are of multiplicity at least $k+1$, and $f^{(k)}g^{(k)}\equiv 1$, then $f(z)=c_1e^{cz}$, $g(z)=c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k c_1c_2c^{2k}=1$.

1 Some Lemmas

For the proof of our results we need the following lemmas.

Lemma 1[1,2] Let f be a nonconstant entire function such that $f^{(k+1)}\not\equiv 0$, k a positive integer, and c a nonzero finite complex number. Then

$$T(r,f)\leqslant N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f^{(k)}-c}\right)-N\left(r,\frac{1}{f^{(k+1)}}\right)+S(r,f)\leqslant N_{k+1}\left(r,\frac{1}{f}\right)+$$
$$\overline{N}\left(r,\frac{1}{f^{(k)}-c}\right)-N_0\left(r,\frac{1}{f^{(k+1)}}\right)+S(r,f)$$

where $N_0(r, 1/f^{(k+1)})$ denotes the counting function which only counts those points such that $f^{(k+1)}=0$ but $f(f^{(k)}-c)\neq 0$.

Lemma 2[1,2] Let f be a nonconstant meromorphic function, and let $a_1(z)$ and $a_2(z)$ be two meromorphic functions such that $T(r, a_i)=S(r,f)$ $(i=1,2)$. Then

$$T(r,f)\leqslant\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f-a_1}\right)+\overline{N}\left(r,\frac{1}{f-a_2}\right)+S(r,f)$$

Lemma 3 Let f and g be two nonconstant entire functions such that $f^{(k+1)}\not\equiv 0$, $g^{(k+1)}\not\equiv 0$, and let m and k be two positive integers. If $E_m(1, f^{(k)})=E_m(1, g^{(k)})$, then one of the following cases must occur:

(i) $$T(r,f)+T(r,g)\leqslant N_{k+1}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f}\right)+N_{k+1}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right)+$$
$$\overline{N}\left(r,\frac{1}{g^{(k)}-1}\right)-N_{1)}\left(r,\frac{1}{f^{(k)}-1}\right)+\overline{N}_{(m+1}\left(r,\frac{1}{f^{(k)}-1}\right)+\overline{N}_{(m+1}\left(r,\frac{1}{g^{(k)}-1}\right)+S(r,f)+S(r,g)$$

(ii) $\dfrac{1}{f^{(k)}-1}=\dfrac{bg^{(k)}+a-b}{g^{(k)}-1}$, where $a(\neq 0)$ and b are two constants.

Proof Set

$$\Psi=\frac{f^{(k+2)}}{f^{(k+1)}}-2\frac{f^{(k+1)}}{f^{(k)}-1}-\frac{g^{(k+2)}}{g^{(k+1)}}+2\frac{g^{(k+1)}}{g^{(k)}-1} \tag{1}$$

Next we consider two cases.

Case 1 $\Psi\equiv 0$. Then by (1) we obtain (ii).

Case 2 $\Psi\not\equiv 0$. Let z_0 be a common simple 1-point of both $f^{(k)}(z)$ and $g^{(k)}(z)$, then by a simple computing, we know $\Psi(z_0)=0$. Thus we have

$$N_{1)}\left(r,\frac{1}{f^{(k)}-1}\right)=N_{1)}\left(r,\frac{1}{g^{(k)}-1}\right)\leqslant N\left(r,\frac{1}{\Psi}\right)\leqslant N(r,\Psi)+S(r,f)+S(r,g) \tag{2}$$

By our assumptions, $\Psi(z)$ have poles only at zeros of $f^{(k+1)}(z)$ and $g^{(k+1)}(z)$. Thus we deduce from (1) that

$$N(r,\Psi)\leqslant\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)+N_0\left(r,\frac{1}{f^{(k+1)}}\right)+N_0\left(r,\frac{1}{g^{(k+1)}}\right)+$$
$$\overline{N}_{(m+1}\left(r,\frac{1}{f^{(k)}-1}\right)+\overline{N}_{(m+1}\left(r,\frac{1}{g^{(k)}-1}\right) \tag{3}$$

where $N_0(r,1/f^{(k+1)})$ denotes the counting function which only counts those points such that $f^{(k+1)}=0$ but $f(f^{(k)}-1)\neq 0$, and $N_0(r,1/g^{(k+1)})$ denotes the analogous quantity.

By lemma 1, we obtain

$$T(r,f)\leqslant N_{k+1}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right)-N_0\left(r,\frac{1}{f^{(k+1)}}\right)+S(r,f) \tag{4}$$

$$T(r,g)\leqslant N_{k+1}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g^{(k)}-1}\right)-N_0\left(r,\frac{1}{g^{(k+1)}}\right)+S(r,g) \tag{5}$$

Combining (2) to (5), we soon obtain that case (i) holds. The proof of lemma 3 is complete.

Lemma 4 Let f and g be two nonconstant entire functions such that $f^{(k+1)}\not\equiv 0$, $g^{(k+1)}\not\equiv 0$, and let k be a positive integer. If $f^{(k)}(z)$ and $g^{(k)}(z)$ share 1 IM, then one of the following cases must occur:

(i) $T(r,f)+T(r,g)\leqslant N_{k+1}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f}\right)+N_{k+1}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g}\right)+\frac{1}{2}N\left(r,\frac{1}{f^{(k)}-1}\right)+$
$\frac{1}{2}N\left(r,\frac{1}{g^{(k)}-1}\right)+\frac{3}{2}\overline{N}_{L}\left(r,\frac{1}{f^{(k)}-1}\right)+\frac{3}{2}\overline{N}_{L}\left(r,\frac{1}{g^{(k)}-1}\right)+S(r,f)+S(r,g)$

(ii) $\frac{1}{f^{(k)}-1}=\frac{bg^{(k)}+a-b}{g^{(k)}-1}$, where $a(\neq 0)$ and b are two constants.

Proof Suppose that Ψ is given by (1). Similarly, if $\Psi\equiv 0$, then we can get (ii) by (1).

Next we consider the case that $\Psi\not\equiv 0$. If z_0 is a common simple 1-point of both $f^{(k)}(z)$ and $g^{(k)}(z)$, then $\Psi(z_0)=0$. Thus we have

$$N_{11}\left(r,\frac{1}{f^{(k)}-1}\right)\leqslant N\left(r,\frac{1}{\Psi}\right)\leqslant N(r,\Psi)+S(r,f)+S(r,g) \tag{6}$$

By computing, we know that the common 1-points of $f^{(k)}$ and $g^{(k)}$ with the same multiplicity are not poles of $\Psi(z)$. Thus we have

$$N(r,\Psi)\leqslant\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)+N_0\left(r,\frac{1}{f^{(k+1)}}\right)+N_0\left(r,\frac{1}{g^{(k+1)}}\right)+$$
$$\overline{N}_{L}\left(r,\frac{1}{f^{(k)}-1}\right)+\overline{N}_{L}\left(r,\frac{1}{g^{(k)}-1}\right) \tag{7}$$

where $N_0(r,1/f^{(k+1)})$ and $N_0(r,1/g^{(k+1)})$ are defined in lemma 3.

Noting that

$$\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right)+\overline{N}\left(r,\frac{1}{g^{(k)}-1}\right)\leqslant N_{11}\left(r,\frac{1}{f^{(k)}-1}\right)+\frac{1}{2}\overline{N}_{L}\left(r,\frac{1}{f^{(k)}-1}\right)+\frac{1}{2}\overline{N}_{L}\left(r,\frac{1}{g^{(k)}-1}\right)+$$
$$\frac{1}{2}N\left(r,\frac{1}{f^{(k)}-1}\right)+\frac{1}{2}N\left(r,\frac{1}{g^{(k)}-1}\right)+S(r,f)+S(r,g) \tag{8}$$

Combining (4) to (8), we soon obtain that case (i) holds. The proof of lemma 4 is complete.

Lemma 5[12] Let f be a nonconstant meromorphic function and k be a positive integer. Then

$$N\left(r,\frac{1}{f^{(k)}}\right)\leqslant N\left(r,\frac{1}{f}\right)+k\overline{N}(r,f)+S(r,f)$$

Lemma 6[13] Let f be a nonconstant entire function and k be a positive integer with $k\geqslant 2$. If $f(z)f^{(k)}(z)\neq 0$, then $f(z)=\mathrm{e}^{az+b}$, where $a(\neq 0)$ and b are two constants.

2 Proof of Proposition 1

Because $f(z)$ and $g(z)$ are two entire functions, by

$$f^{(k)}(z)g^{(k)}(z)=1 \tag{9}$$

we have $f^{(k)}(z)\neq 0$ and $g^{(k)}(z)\neq 0$.

Let z_0 be a zero of f with multiplicity n. Then z_0 is a pole of $g^{(k)}$ with multiplicity $n-k(n\geqslant k+1)$, which contradicts that g is an entire function. Hence $f(z)\neq 0$. Similarly $g(z)\neq 0$.

If $k\geqslant 2$, then by $f(z)f^{(k)}(z)\neq 0$, $g(z)g^{(k)}(z)\neq 0$ and lemma 6, we deduce from (9) that $f(z)=c_1e^{cz}$, $g(z)=c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k c_1c_2c^{2k}=1$.

If $k=1$, then by $f(z)\neq 0$ and $g(z)\neq 0$, there exist two entire functions $\alpha(z)$ and $\beta(z)$ such that $f(z)=e^{\alpha(z)}$, $g(z)=e^{\beta(z)}$. Therefore we have $f'(z)=\alpha'(z)e^{\alpha(z)}\neq 0$, $g'(z)=\beta'(z)e^{\beta(z)}\neq 0$, that is $\alpha'(z)\neq 0$, $\beta'(z)\neq 0$. Thus $\alpha'(z)=e^{\gamma(z)}$, $\beta'(z)=e^{\delta(z)}$, where $\gamma(z)$ and $\delta(z)$ are two entire functions.

From (9) we get

$$e^{\alpha(z)+\beta(z)+\gamma(z)+\delta(z)}=1$$

Differentiating the above equation gives

$$(\alpha'(z)+\beta'(z)+\gamma'(z)+\delta'(z))\ e^{\alpha(z)+\beta(z)+\gamma(z)+\delta(z)}\equiv 0$$

Thus

$$\alpha'(z)+\beta'(z)+\gamma'(z)+\delta'(z)\equiv 0$$

that is

$$e^{\gamma(z)}+\gamma'(z)\equiv -(e^{\delta(z)}+\delta'(z))$$

By

$$T(r,\ \gamma'(z))=m(r,\ \gamma'(z))=m\left(r,\ \frac{(e^{\gamma(z)})'}{e^{\gamma(z)}}\right)=S(r,\ e^{\gamma(z)})$$

$$T(r,\ \delta'(z))=m(r,\ \delta'(z))=m\left(r,\ \frac{(e^{\delta(z)})'}{e^{\delta(z)}}\right)=S(r,\ e^{\delta(z)})$$

we have

$$T(r,\ e^{\gamma(z)})=T(r,\ e^{\delta(z)})+S(r,\ e^{\gamma(z)})$$

Thus

$$e^{\gamma(z)}+e^{\delta(z)}\equiv -(\gamma'(z)+\delta'(z))=a(z) \tag{10}$$

where $a(z)$ is a common small function of both $e^{\gamma(z)}$ and $e^{\delta(z)}$.

Suppose that $a(z)\not\equiv 0$, then $e^{\gamma(z)}/a(z)+e^{\delta(z)}/a(z)\equiv 1$. By the Nevanlinna second fundamental theorem, we obtain

$$T(r,\ e^{\delta(z)})=T\left(r,\ \frac{e^{\delta(z)}}{a(z)}\right)+S(r,\ e^{\delta(z)})\leqslant\overline{N}\left(r,\ \frac{e^{\delta(z)}}{a(z)}\right)+\overline{N}\left(r,\ \frac{1}{e^{\delta(z)}/a(z)}\right)+\overline{N}\left(r,\ \frac{1}{e^{\delta(z)}/a(z)-1}\right)+$$

$$S(r,\ e^{\delta(z)})=\overline{N}\left(r,\ \frac{1}{e^{\delta(z)}/a(z)-1}\right)+S(r,\ e^{\delta(z)})=\overline{N}\left(r,\ \frac{1}{e^{\gamma(z)}/a(z)}\right)+S(r,\ e^{\delta(z)})=S(r,\ e^{\delta(z)})$$

Thus we deduce that $e^{\delta(z)}$ is a constant. Similarly $e^{\gamma(z)}$ is a constant. Hence $\delta(z)$ and $\gamma(z)$ are also two constants, and $\delta'(z)=0$, $\gamma'(z)=0$. Then $\delta'(z)+\gamma'(z)\equiv 0$. From (10) we get $a(z)\equiv 0$, a contradiction.

Thus

$$\delta'(z)+\gamma'(z)\equiv 0,\ e^{\gamma(z)}+e^{\delta(z)}=\alpha'(z)+\beta'(z)\equiv 0$$

and

$$\gamma(z)+\delta(z)=b,\ e^{\gamma(z)}+e^{b-\gamma(z)}\equiv 0$$

where b is a constant. Hence we get

$$\gamma(z)\equiv c_1,\ \delta(z)=b-\gamma(z)\equiv c_2$$

$$\alpha'(z)=e^{\gamma(z)}\equiv c,\ \beta'(z)=-\alpha'(z)\equiv -c$$

where c_1, c_2 and c are three constants.

Let

$$\alpha(z)=cz+\ln c_1,\ \beta(z)=-cz+\ln c_2$$

then $e^{\alpha(z)}=c_1e^{cz}$, $e^{\beta(z)}=c_2e^{-cz}$, that is $f(z)=c_1e^{cz}$, $g(z)=c_2e^{-cz}$. Substituting it into (9) we soon obtain

$(c_1c_2)c^2 = -1$. The proof of proposition 1 is complete.

3 Proof of Theorem 7

By $m \geqslant 3$ we have

$$\overline{N}\left(r, \frac{1}{f^{(k)}-1}\right) - \frac{1}{2}N_{1)}\left(r, \frac{1}{f^{(k)}-1}\right) + \overline{N}_{(m+1}\left(r, \frac{1}{f^{(k)}-1}\right) \leqslant \frac{1}{2N}\left(r, \frac{1}{f^{(k)}-1}\right)$$

$$\overline{N}\left(r, \frac{1}{g^{(k)}-1}\right) - \frac{1}{2}N_{1)}\left(r, \frac{1}{g^{(k)}-1}\right) + \overline{N}_{(m+1}\left(r, \frac{1}{g^{(k)}-1}\right) \leqslant \frac{1}{2}N\left(r, \frac{1}{g^{(k)}-1}\right)$$

By the Nevanlinna first fundamental theorem, we have

$$N\left(r, \frac{1}{f^{(k)}-1}\right) \leqslant T(r, f^{(k)}) + O(1) \leqslant T(r, f) + S(r, f)$$

$$N\left(r, \frac{1}{f}\right) \leqslant T(r, f) + S(r, f)$$

Similarly, we have

$$N\left(r, \frac{1}{g^{(k)}-1}\right) \leqslant T(r, g) + S(r, g)$$

$$N\left(r, \frac{1}{g}\right) \leqslant T(r, g) + S(r, g)$$

Obviously,

$$\overline{N}\left(r, \frac{1}{f}\right) \leqslant \frac{1}{n}N\left(r, \frac{1}{f}\right), \quad N_{k+1}\left(r, \frac{1}{f}\right) \leqslant (k+1)\overline{N}\left(r, \frac{1}{f}\right) \leqslant \frac{k+1}{n}N\left(r, \frac{1}{f}\right)$$

$$\overline{N}\left(r, \frac{1}{g}\right) \leqslant \frac{1}{n}N\left(r, \frac{1}{g}\right), \quad N_{k+1}\left(r, \frac{1}{g}\right) \leqslant (k+1)\overline{N}\left(r, \frac{1}{g}\right) \leqslant \frac{k+1}{n}N\left(r, \frac{1}{g}\right)$$

Suppose that

$$T(r, f) + T(r, g) \leqslant N_{k+1}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f}\right) + N_{k+1}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{f^{(k)}-1}\right) +$$
$$\overline{N}\left(r, \frac{1}{g^{(k)}-1}\right) - N_{1)}\left(r, \frac{1}{f^{(k)}-1}\right) + \overline{N}_{(m+1}\left(r, \frac{1}{f^{(k)}-1}\right) + \overline{N}_{(m+1}\left(r, \frac{1}{g^{(k)}-1}\right) + S(r, f) + S(r, g)$$

then we have

$$T(r, f) + T(r, g) \leqslant \left(\frac{1}{2} + \frac{k+2}{n}\right)T(r, f) + \left(\frac{1}{2} + \frac{k+2}{n}\right)T(r, g) + S(r, f) + S(r, g)$$

That is

$$\left(\frac{1}{2} - \frac{k+2}{n}\right) + \left(\frac{1}{2} - \frac{k+2}{n}\right) \leqslant S(r, f) + S(r, g)$$

It gives $T(r, f) + T(r, g) \leqslant S(r, f) + S(r, g)$ because $n > 2k+4$. This is impossible. Hence by lemma 3 we have

$$\frac{1}{f^{(k)}-1} = \frac{bg^{(k)} + a - b}{g^{(k)}-1} \tag{11}$$

where $a(\neq 0)$ and b are two constants.

Next we consider three cases.

Case 1 $b \neq 0$, $a = b$. Then (11) becomes $1/(f^{(k)} - 1) = bg^{(k)}/(g^{(k)} - 1)$.

If $b = -1$, then $f^{(k)}g^{(k)} = 1$. By proposition 1, we obtain that $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants with $(-1)^k c_1 c_2 c^{2k} = 1$.

If $b \neq -1$, then $f^{(k)} - (1 + 1/b) = -1/(bg^{(k)}) \neq 0$. By lemma 1, we have

$$T(r, f) \leqslant N_{k+1}\left(r, \frac{1}{f}\right) + S(r, f) \leqslant \frac{k+1}{n}N\left(r, \frac{1}{f}\right) + S(r, f) \leqslant \frac{k+1}{n}T(r, f) + S(r, f)$$

that is

$$\left(1 - \frac{k+1}{n}\right)T(r, f) \leqslant S(r, f)$$

It gives $T(r, f) \leqslant S(r, f)$ because $n > 2k+4$. This is impossible.

Case 2 $b \neq 0$, $a \neq 0$. From (11) we get $g^{(k)} + (a-b)/b \neq 0$. Thus by lemma 1 we have

$$T(r, g) \leqslant N_{k+1}\left(r, \frac{1}{g}\right) + S(r, g)$$

Similarly we can deduce a contradiction as in case 1.

Case 3 $b=0$, $a\neq 0$. Then by (11) we have

$$f^{(k)}=\frac{1}{a}g^{(k)}+1-\frac{1}{a} \tag{12}$$

Thus we get

$$f=\frac{1}{a}g+\left(1-\frac{1}{a}\right)\frac{1}{k!}z^k+p(z) \tag{13}$$

where $p(z)$ is a polynomial of degree at most $k-1$.

Now we prove that $q(z)=(1-1/a)z^k/k!+p(z)\equiv 0$. We consider two cases.

(i) f and g are two transcendental entire functions. Suppose that $q(z)\not\equiv 0$, then by lemma 2 we have

$$T(r,f)\leqslant \overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-q}\right)+S(r,f)=\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)+S(r,f)\leqslant$$
$$\frac{1}{n}N\left(r,\frac{1}{f}\right)+\frac{1}{n}N\left(r,\frac{1}{g}\right)+S(r,f)\leqslant\frac{1}{n}T(r,f)+\frac{1}{n}T(r,g)+S(r,f) \tag{14}$$

Obviously, we get from (13) that $T(r,f)=T(r,g)+S(r,f)$. Substituting it into (14), we get $(1-2/n)\cdot T(r,f)\leqslant S(r,f)$. It gives $T(r,f)\leqslant S(r,f)$ because $n>2k+4$. This is impossible. Thus $(1-1/a)z^k/k!+p(z)\equiv 0$. Hence by (13), we obtain $f\equiv g$.

(ii) f and g are two polynomials.

Let f and g have s and t pairwise distinct zeros, respectively, $f(z)=c_1(z-a_1)^{l_1}(z-a_2)^{l_2}\cdots(z-a_s)^{l_s}$, $g(z)=c_2(z-b_1)^{m_1}(z-b_2)^{m_2}\cdots(z-b_t)^{m_t}$, where c_1 and c_2 are two constants, $l_i>2k+4\,(i=1,2,\cdots,s)$, $m_j>2k+4\ (j=1,2,\cdots,t)$. Differentiating the two sides of (12), we get $f^{(k+1)}=g^{(k+1)}/a$. Hence we have

$$(z-a_1)^{l_1-(k+1)}\cdots(z-a_s)^{l_s-(k+1)}p_1(z)=(z-b_1)^{m_1-(k+1)}\cdots(z-b_t)^{m_t-(k+1)}p_2(z) \tag{15}$$

where $p_1(z)$ and $p_2(z)$ are two polynomials with $\deg p_1=(s-1)(k+1)$, $\deg p_2=(t-1)(k+1)$.

By $l_i>2k+4\,(i=1,2,\cdots,s)$, $m_j>2k+4\,(j=1,2,\cdots,t)$, we have $\sum_{i=1}^{s}(l_i-(k+1))>s(k+3)>(s-1)(k+1)$, $\sum_{j=1}^{t}(m_j-(k+1))>t(k+3)>(t-1)(k+1)$. Then from (15), we know that there exists z_0 such that $f(z_0)=g(z_0)=0$. From (13) and $f(z_0)=g(z_0)=0$ with the multiplicity of $z_0>2k+4$, we get $(1-1/a)z^k/k!+p(z)\equiv 0$. Thus we obtain $f\equiv g$. The proof of theorem 7 is complete.

4 Proof of Theorem 5 and Theorem 6

First we prove theorem 3.

Obviously,

$$\left(N\left(r,\frac{1}{f^{(k)}-1}\right)-\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right)\right)+\left(N\left(r,\frac{1}{f^{(k)}}\right)-\overline{N}\left(r,\frac{1}{f^{(k)}}\right)\right)\leqslant N\left(r,\frac{1}{f^{(k+1)}}\right)$$

From this and lemma 5, we obtain

$$N\left(r,\frac{1}{f^{(k)}-1}\right)-\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right)\leqslant\overline{N}\left(r,\frac{1}{f^{(k)}}\right)+S(r,f) \tag{16}$$

Noting that

$$\overline{N}\left(r,\frac{1}{f^{(k)}}\right)=N\left(r,\frac{1}{f^{(k)}}\right)-\left(N\left(r,\frac{1}{f^{(k)}}\right)-\overline{N}\left(r,\frac{1}{f^{(k)}}\right)\right)$$

then by lemma 5, we get

$$\overline{N}\left(r,\frac{1}{f^{(k)}}\right)\leqslant N\left(r,\frac{1}{f}\right)-\left(N\left(r,\frac{1}{f^{(k)}}\right)-\overline{N}\left(r,\frac{1}{f^{(k)}}\right)\right)+S(r,f)\leqslant N_{k+1}\left(r,\frac{1}{f}\right)+S(r,f)\leqslant$$
$$\frac{k+1}{n}N\left(r,\frac{1}{f}\right)+S(r,f) \tag{17}$$

Thus from (16) and (17), we have

$$\overline{N}_{(3}\left(r,\frac{1}{f^{(k)}-1}\right)\leqslant\frac{1}{2}\left(N\left(r,\frac{1}{f^{(k)}-1}\right)-\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right)\right)\leqslant\frac{1}{2}\overline{N}\left(r,\frac{1}{f^{(k)}}\right)+S(r,f)\leqslant$$
$$\frac{k+1}{2n}N\left(r,\frac{1}{f}\right)+S(r,f)$$

Similarly, we have

$$\overline{N}_{(3}\left(r,\frac{1}{g^{(k)}-1}\right)\leqslant\frac{k+1}{2n}N\left(r,\frac{1}{g}\right)+S(r,g)$$

Obviously,

$$\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right)-\frac{1}{2}N_{1)}\left(r,\frac{1}{f^{(k)}-1}\right)+\frac{1}{2}\overline{N}_{(3}\left(r,\frac{1}{f^{(k)}-1}\right)\leqslant\frac{1}{2}N\left(r,\frac{1}{f^{(k)}-1}\right)$$

$$\overline{N}\left(r,\frac{1}{g^{(k)}-1}\right)-\frac{1}{2}N_{1)}\left(r,\frac{1}{g^{(k)}-1}\right)+\frac{1}{2}\overline{N}_{(3}\left(r,\frac{1}{g^{(k)}-1}\right)\leqslant\frac{1}{2}N\left(r,\frac{1}{g^{(k)}-1}\right)$$

Suppose that

$$T(r,f)+T(r,g)\leqslant N_{k+1}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f}\right)+N_{k+1}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right)+$$
$$\overline{N}\left(r,\frac{1}{g^{(k)}-1}\right)-N_{1)}\left(r,\frac{1}{f^{(k)}-1}\right)+\overline{N}_{(3}\left(r,\frac{1}{f^{(k)}-1}\right)+\overline{N}_{(3}\left(r,\frac{1}{g^{(k)}-1}\right)+S(r,f)+S(r,g)$$

then we obtain that

$$T(r,f)+T(r,g)\leqslant\left(\frac{1}{2}+\frac{5k+9}{4n}\right)T(r,f)+\left(\frac{1}{2}+\frac{5k+9}{4n}\right)T(r,g)+S(r,f)+S(r,g)$$

That is

$$\left(\frac{1}{2}-\frac{5k+9}{4n}\right)T(r,f)+\left(\frac{1}{2}-\frac{5k+9}{4n}\right)\leqslant S(r,f)+S(r,g)$$

It gives $T(r,f)+T(r,g)\leqslant S(r,f)+S(r,g)$ because $n>(5k+9)/2$. This is impossible. Thus by lemma 3, we have

$$\frac{1}{f^{(k)}-1}=\frac{bg^{(k)}+a-b}{g^{(k)}-1}$$

where $a(\neq 0)$ and b are two constants.

In the following we complete the proof of theorem 6 as done in theorem 7. Theorem 6 is proved.

Now we prove theorem 5.

From (16) and (17), we have

$$\overline{N}_{(2}\left(r,\frac{1}{f^{(k)}-1}\right)\leqslant N\left(r,\frac{1}{f^{(k)}-1}\right)-\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right)\leqslant\overline{N}\left(r,\frac{1}{f^{(k)}}\right)+S(r,f)\leqslant\frac{k+1}{n}N\left(r,\frac{1}{f}\right)+S(r,f)$$

Similarly, we have

$$\overline{N}_{(2}\left(r,\frac{1}{g^{(k)}-1}\right)\leqslant\frac{k+1}{n}N\left(r,\frac{1}{g}\right)+S(r,g)$$

Obviously,

$$\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right)-\frac{1}{2}N_{1)}\left(r,\frac{1}{f^{(k)}-1}\right)\leqslant\frac{1}{2}N\left(r,\frac{1}{f^{(k)}-1}\right)$$

$$\overline{N}\left(r,\frac{1}{g^{(k)}-1}\right)-\frac{1}{2}N_{1)}\left(r,\frac{1}{g^{(k)}-1}\right)\leqslant\frac{1}{2}N\left(r,\frac{1}{g^{(k)}-1}\right)$$

Suppose that

$$T(r,f)+T(r,g)\leqslant N_{k+1}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f}\right)+N_{k+1}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right)+$$
$$\overline{N}\left(r,\frac{1}{g^{(k)}-1}\right)-N_{1)}\left(r,\frac{1}{f^{(k)}-1}\right)+\overline{N}_{(2}\left(r,\frac{1}{f^{(k)}-1}\right)+\overline{N}_{(2}\left(r,\frac{1}{g^{(k)}-1}\right)+S(r,f)+S(r,g)$$

then we obtain that

$$\left(\frac{1}{2}-\frac{2k+3}{n}\right)T(r,f)+\left(\frac{1}{2}-\frac{2k+3}{n}\right)T(r,g)\leqslant S(r,f)+S(r,g)$$

It gives $T(r,f)+T(r,g)\leqslant S(r,f)+S(r,g)$ because $n>4k+6$. Thus by lemma 3 we have

$$\frac{1}{f^{(k)}-1}=\frac{bg^{(k)}+a-b}{g^{(k)}-1}$$

where $a(\neq 0)$ and b are two constants.

Next we complete the proof of theorem 5 as done in theorem 7. Theorem 5 is proved.

5 Proof of Theorem 8

From (16) and (17), we have

$$\overline{N}_{L}\left(r,\frac{1}{f^{(k)}-1}\right)\leqslant N\left(r,\frac{1}{f^{(k)}-1}\right)-\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right)\leqslant\overline{N}\left(r,\frac{1}{f^{(k)}}\right)+S(r,f)\leqslant\frac{k+1}{n}N\left(r,\frac{1}{f}\right)+S(r,f)$$

Similarly, we have

$$\overline{N}_{\mathrm{L}}\left(r, \frac{1}{g^{(k)}-1}\right) \leqslant \frac{k+1}{n} N\left(r, \frac{1}{g}\right) + S(r, g)$$

Suppose that

$$T(r, f) + T(r, g) \leqslant N_{k+1}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f}\right) + N_{k+1}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g}\right) + \frac{1}{2} N\left(r, \frac{1}{f^{(k)}-1}\right) + \frac{1}{2} N\left(r, \frac{1}{g^{(k)}-1}\right) + \frac{3}{2}\overline{N}_{\mathrm{L}}\left(r, \frac{1}{f^{(k)}-1}\right) + \frac{3}{2}\overline{N}_{\mathrm{L}}\left(r, \frac{1}{g^{(k)}-1}\right) + S(r, f) + S(r, g)$$

then we obtain that

$$\left(\frac{1}{2} - \frac{5k+7}{2n}\right) T(r, f) + \left(\frac{1}{2} - \frac{5k+7}{2n}\right) T(r, g) \leqslant S(r, f) + S(r, g)$$

It gives $T(r, f) + T(r, g) \leqslant S(r, f) + S(r, g)$ because $n > 5k+7$. This is impossible. Thus by lemma 4 we have

$$\frac{1}{f^{(k)}-1} = \frac{bg^{(k)} + a - b}{g^{(k)}-1}$$

where $a(\neq 0)$ and b are two constants.

Next we complete the proof of theorem 8 as done in theorem 7. Theorem 8 is proved.

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整函数的惟一性

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摘要: 研究了涉及导函数的整函数的惟一性, 主要证明了以下结果. 设 $f(z)$ 和 $g(z)$ 为非常数整函数, n, k 为满足 $n > 2k+4$ 的 2 个正整数. 若$f(z)$和$g(z)$的零点重数均至少为 n, 且$f^{(k)}(z)$和$g^{(k)}(z)$ CM 分担 1, 则或者$f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, 其中 c_1, c_2和 c 为满足 $(-1)^k c_1 c_2 c^{2k} = 1$ 的常数; 或者$f(z) \equiv g(z)$.

关键词: 整函数; 分担值; 惟一性

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