Journal of Southeast University (English Edition) Vol. 20 No. 4 Dec. 2004 ISSN 1003—7985

Weight-based shape modification of NURBS curves by constrained optimization

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Abstract: A new method for shape modification of non-uniform rational B-splines (NURBS) curves is presented, which is based on constrained optimization by means of altering the corresponding weights of their control points. Using this method, the original NURBS curve is modified to satisfy the specified geometric constraints, including single point and multi-point constraints. With the introduction of free parameters, the shapes of modified NURBS curves can be further controlled by users without destroying geometric constraints and seem more naturally. Since explicit formulae are derived to compute new weights of the modified curve, the method is simple and easy to program. Practical examples show that the method is applicable for computer aided design (CAD) system.

Key words: non-uniform rational B-splines; shape modification; constrained optimization; convex hull

The non-uniform rational B-splines (NURBS) have become the de-facto standard for designing or representing all kinds of complicated shapes in the field of computer aided geometric design (CAGD) and computer graphics because they give a uniform description of free-form shapes and analytical curves or surfaces. Some complex geometric models might not have been created without NURBS as they can represent ordinary B-splines and Bézier as well as all conic curves. However, shape design is a time-consuming process and usually cannot be accomplished in one stroke. After creating NURBS curves or surfaces, we often need to modify them such that their shapes satisfy our design requirements.

Shape modification of parametric curves and surfaces has been attentively investigated. Piegl[1] proposed two classic approaches to modifying the shape of an NURBS curve and surface: control-point-based modification and weight-based modification. Au and Yuen[2] presented an approach based on altering the weights and locations of the control points simultaneously to deform the NURBS curve. Fowler and Bartels[3] investigated a method to let users interactively specify properties that depended on the curve's values and derivatives at selected parametric points. Sánchez[4] developed a simple technique to modify NURBS curves based on a perspective functional transformation of arbitrary origin. Hu, et al.[5,6] presented a method for shape modification of NURBS surfaces via geometric constraint optimization.

Physically-based shape modifications also have been developed during the past twenty years. In the physically-based method, curves and surfaces are viewed as real objects that are subject to physical laws, and can be allowed to stretch and bend. Terzopoulos[7] proposed elastically deformable models of surface and derived different deformation equations by means of elastic theory in 1987. Celniker and Gossard[8] investigated a method to deform constrained B-spline curves and surfaces by minimizing deformable energy. Welch and Witkin[9] proposed variational surfaces in geometric modeling and deformed their shapes by constraint optimization. In brief, in a physically-based method we need to compute the surfaces satisfying specified constraints with minimal deformable energy.

However, in some cases, we only need to deform the original curve in a subtle area without moving control points, therefore altering weights of the curve is a good choice. In this paper, we present a new method to modify the shape of NURBS curves by minimizing the variation of weights by means of the least square method. Explicit formulae are also derived to compute new weights of the modified curve.

1 Problem Statement

An NURBS curve with control points $\boldsymbol{P}_i$ $(0 \leqslant i \leqslant n)$ can be defined as

$$\boldsymbol{P}(u) = \frac{\sum_{i=0}^{n} \omega_i \boldsymbol{P}_i N_{i,k}(u)}{\sum_{i=0}^{n} \omega_i N_{i,k}(u)} \qquad u_k \leqslant u \leqslant u_{n+1} \quad (1)$$

Received 2004-05-14.

Foundation item: The Teaching and Research Award Program for Outstanding Young Teachers in Higher Education Institutions of MOE, P. R. C.

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where ω_i are corresponding weights of control points and $N_{i,k}(u)$ are the normalized B-spline base functions of degree k defined over knot vector $\boldsymbol{U} = \{u_0, \cdots, u_k, \cdots, u_{n+1}, \cdots, u_{n+k+1}\}$.

As shown in Fig.1, suppose $\boldsymbol{T}$ is a target point in modified curve $\hat{\boldsymbol{P}}(u)$ and $\boldsymbol{S}$ is a start point in original curve $\boldsymbol{P}(u)$ with the same parameter $\hat{u} \in [u_m, u_{m+1}]$. How should we modify original curve such that the modified curve passes through $\boldsymbol{T}$ exactly?

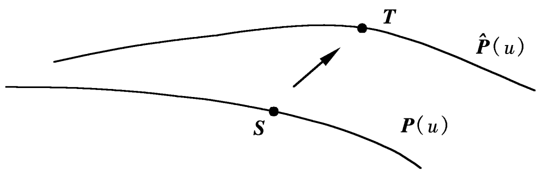


Fig.1 Illustration of shape modification

2 Constrained Optimization Solution for Single Point Constraint

2.1 Single target point constraint

An NURBS curve can also be rewritten as

$$\boldsymbol{P}(u) = \frac{1}{W(u)} \sum_{i=0}^{n} \omega_i \boldsymbol{P}_i N_{i,k}(u) \tag{2}$$

where $W(u) = \sum_{i=0}^{n} \omega_i N_{i,k}(u)$. Suppose that weights $\omega_j (j_1 \leqslant j \leqslant j_2,\ m-k \leqslant j \leqslant m)$ are to be changed, and we choose perturbation $\Delta\omega_j\ (j_1 \leqslant j \leqslant j_2)$ for those weights $(\omega_j^* = \omega_j + \Delta\omega_j)$. As a result, the modified curve takes the form of

$$\hat{\boldsymbol{P}}(u) = \frac{\sum_{i=0}^{n} \omega_i \boldsymbol{P}_i N_{i,k}(u) + \sum_{j=j_1}^{j_2} \Delta\omega_j \boldsymbol{P}_j N_{j,k}(u)}{W(u) + \sum_{j=j_1}^{j_2} \Delta\omega_j N_{j,k}(u)} \tag{3}$$

For the start point, we can get the following equation:

$$W(\hat{u})\boldsymbol{S} = \sum_{i=0}^{n} \omega_i \boldsymbol{P}_i N_{i,k}(\hat{u}) \tag{4}$$

For the target point, Eq.(5) can be derived.

$$W(\hat{u})\boldsymbol{T} + \boldsymbol{T}\sum_{j=j_1}^{j_2} \Delta\omega_j N_{j,k}(\hat{u}) = \sum_{i=0}^{n} \omega_i \boldsymbol{P}_i N_{i,k}(\hat{u}) + \sum_{j=j_1}^{j_2} \Delta\omega_j \boldsymbol{P}_j N_{j,k}(\hat{u}) \tag{5}$$

From Eq.(4) and Eq.(5), we have

$$(\boldsymbol{T} - \boldsymbol{S})W(\hat{u}) = \sum_{j=j_1}^{j_2} \Delta\omega_j N_{j,k}(\hat{u})(\boldsymbol{P}_j - \boldsymbol{T}) \tag{6}$$

We determine $\Delta\omega_j\ (j = j_1, \cdots, j_2)$ by the constrained optimization method. The optimization objective function is $\sum_{j=j_1}^{j_2} \alpha_j \Delta\omega_j^2 = \min$, where $\alpha_j\ (j = j_1, \cdots, j_2;\ 0 < \alpha_j < 1)$ are free parameters which are given by designers. Here we set $\sum_{j=j_1}^{j_2} \alpha_j = 1$. Consequently, the Lagrange function is defined as

$$L = \sum_{j=j_1}^{j_2} \alpha_j \Delta\omega_j^2 + \boldsymbol{K}\left[(\boldsymbol{T} - \boldsymbol{S})W(\hat{u}) - \sum_{j=j_1}^{j_2} \Delta\omega_j N_{j,k}(\hat{u})(\boldsymbol{P}_j - \boldsymbol{T})\right] \tag{7}$$

where $\boldsymbol{K} = \{k_x, k_y\}^{\mathrm{T}}$ is a Lagrange multiplier.

Let $\frac{\partial}{\partial k_x}(L) = \frac{\partial}{\partial k_y}(L) = 0$ and $\frac{\partial}{\partial \Delta\omega_j}(L) = 0$ for $j = j_1, \cdots, j_2$, and write derived formulae in vector form, then the following equations can be derived.

$$\left.\begin{aligned} (\boldsymbol{T} - \boldsymbol{S})W(\hat{u}) &= \sum_{j=j_1}^{j_2} \Delta\omega_j N_{j,k}(\hat{u})(\boldsymbol{P}_j - \boldsymbol{T}) \\ 2\alpha_j \Delta\omega_j &= N_{j,k}(\hat{u})\boldsymbol{K}(\boldsymbol{P}_j - \boldsymbol{T}) \end{aligned}\right\} \tag{8}$$

From Eq.(8), Eq.(9) can be derived.

$$\Delta\omega_j = \frac{N_{j,k}(\hat{u})(\boldsymbol{P}_j - \boldsymbol{T})^{\mathrm{T}}\boldsymbol{K}}{2\alpha_j} \tag{9}$$

Therefore, we have

$$(\boldsymbol{T} - \boldsymbol{S})W(\hat{u}) = \frac{1}{2}\sum_{j=j_1}^{j_2} \frac{N_{j,k}^2(\hat{u})}{\alpha_j} \cdot (\boldsymbol{P}_j - \boldsymbol{T})(\boldsymbol{P}_j - \boldsymbol{T})^{\mathrm{T}}\boldsymbol{K} \tag{10}$$

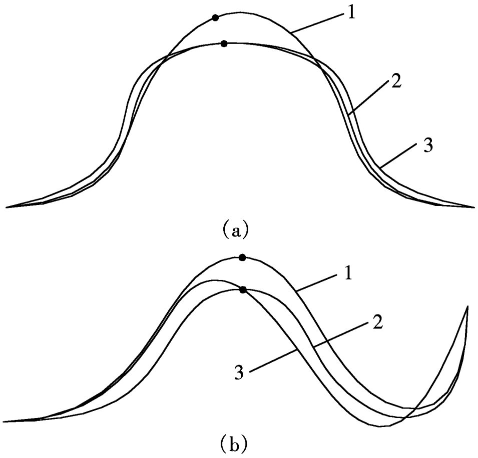
Setting $\boldsymbol{A}_j = \frac{N_{j,k}^2(\hat{u})}{\alpha_j}(\boldsymbol{P}_j - \boldsymbol{T})(\boldsymbol{P}_j - \boldsymbol{T})^{\mathrm{T}}$ for $j_1 \leqslant j \leqslant j_2$ and $\boldsymbol{A} = \sum_{j=j_1}^{j_2} \boldsymbol{A}_j$ where $\boldsymbol{A}_j$ is a 2×2 matrix, we can get explicit solution for single point constraint from Eqs.(8), (9) and (10).

$$\Delta\omega_j = \frac{N_{j,k}(\hat{u})W(\hat{u})}{\alpha_j}(\boldsymbol{P}_j - \boldsymbol{T})^{\mathrm{T}}\boldsymbol{A}^{-1}(\boldsymbol{T} - \boldsymbol{S}) \tag{11}$$

Fig. 2 are examples of single target point constraint. The original curve is an NURBS curve with seven control points. Weights $\omega_i\ (i = 2, 3, 4)$ are changed.

From Fig.2, we can see that free parameters can be adjusted to keep some parts of the original curve less changed. Since negative weights are not allowed, one rule must be emphasized here. When we choose the target point, it must be within the piecewise convex hull, otherwise the shape of the modified curve will be inestimable. We can see that from Fig.3.

In Fig.3, since the target point is not within the piecewise convex hull constructed by control vertices $\boldsymbol{P}_i\ (i = 1, 2, 3, 4)$, the new weight $(\omega_4^* = \omega_4 + \Delta\omega_4)$ equals $-0.167\,3$.



1 — Original NURBS curve; 2 — Modified curve, $\alpha_2 = 0.3$, $\alpha_3 = 0.4$, $\alpha_4 = 0.3$; 3 — Modified curve, $\alpha_2 = 0.01$, $\alpha_3 = 0.98$, $\alpha_4 = 0.01$

Fig.2 Shape modification with single target point constraint

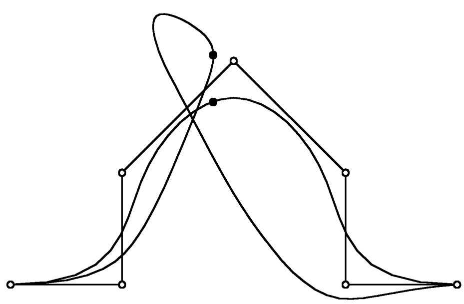


Fig.3 Shape of modified curve with negative weight

2.2 Discussions in some special cases

Eq.(6) accounts for the relationship between the varied weights and the target point. Although we have derived an explicit solution for a single target point constraint, there are some special cases.

Case 1 If only one weight of the control point $\boldsymbol{P}_j$ is changed, Eq.(6) can be rewritten as

$$(\boldsymbol{T}-\boldsymbol{S})W(\hat{u}) = \Delta\omega_j N_{j,k}(\hat{u})(\boldsymbol{P}_j - \boldsymbol{T}) \tag{12}$$

Therefore, only when vector $(\boldsymbol{T}-\boldsymbol{S}) \times (\boldsymbol{P}_j - \boldsymbol{T}) = 0$, that is to say, the target point should be restricted to the line joining the start point $\boldsymbol{S}$ and the control point $\boldsymbol{P}_j$, we can get a unique solution.

$$\Delta\omega_j = \frac{W(\hat{u})(\boldsymbol{P}_j - \boldsymbol{T})^{\mathrm{T}}(\boldsymbol{T}-\boldsymbol{S})}{N_{j,k}(\hat{u})\|\boldsymbol{P}_j - \boldsymbol{T}\|^2} \tag{13}$$

Fig.4 shows the example of shape modification via altering one weight.

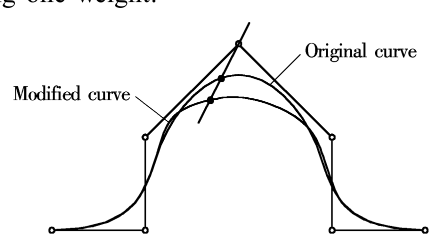


Fig.4 Shape modification via altering one weight

Case 2 If only two weights of control points $\boldsymbol{P}_{j_1}$ and $\boldsymbol{P}_{j_2}$ are changed, Eq.(6) takes the following form:

$$(\boldsymbol{T}-\boldsymbol{S})W(\hat{u}) = \Delta\omega_{j_1} N_{j_1,k}(\hat{u})(\boldsymbol{P}_{j_1} - \boldsymbol{T}) + \Delta\omega_{j_2} N_{j_2,k}(\hat{u})(\boldsymbol{P}_{j_2} - \boldsymbol{T}) \tag{14}$$

When the target point $\boldsymbol{T}$ is not on line 1, line 2 and line 3 (see Fig.5), we only need to solve a linear equation system without optimization.

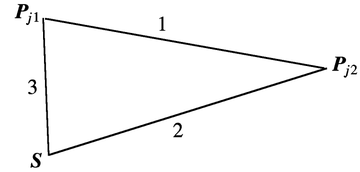


Fig.5 Illustration of start point and two control points

Setting $\boldsymbol{A} = [N_{j_1,k}(\hat{u})(\boldsymbol{P}_{j_1} - \boldsymbol{T}) \quad N_{j_2,k}(\hat{u})(\boldsymbol{P}_{j_2} - \boldsymbol{T})]^{\mathrm{T}}_{2\times 2}$ and $\Delta\boldsymbol{\omega} = \{\Delta\omega_{j_1}, \Delta\omega_{j_2}\}$ where $\boldsymbol{A}$ is a 2×2 matrix, we can obtain a unique solution as

$$\Delta\boldsymbol{\omega} = W(\hat{u})\boldsymbol{A}^{-1}(\boldsymbol{T}-\boldsymbol{S}) \tag{15}$$

When the target point is on line 2 or line 3, we can obtain one solution (such as $\Delta\omega_{j_1}$) from Eq.(13), and the other perturbation ($\Delta\omega_{j_2}$) equals 0.

However, when the target point is restricted to line 1, since matrix $\boldsymbol{A}$ is singular, no solution can be obtained from the constraint equation.

3 Constrained Optimization Solution for Multi-Point Constraint

Suppose there are p target points $\boldsymbol{T}_l$ with parameters $\hat{u}_l$ ($l = 0, 1, \cdots, p$), then the modified curve satisfies the following requirements:

$$(\boldsymbol{T}_l - \boldsymbol{S}_l)W(\hat{u}_l) = \sum_{j=j_1}^{j_2} \Delta\omega_j N_{j,k}(\hat{u}_l)(\boldsymbol{P}_j - \boldsymbol{T}_l) \tag{16}$$

So the Lagrange function is defined as

$$L = \sum_{j=j_1}^{j_2} \alpha_j \Delta\omega_j^2 + \sum_{l=0}^{p} \boldsymbol{k}_l \left[(\boldsymbol{T}_l - \boldsymbol{S}_l)W(\hat{u}_l) - \sum_{j=j_1}^{j_2} \Delta\omega_j N_{j,k}(\hat{u}_l)(\boldsymbol{P}_j - \boldsymbol{T}_l) \right]$$

As a result, the following equations can be derived.

$$\left.\begin{aligned} (\boldsymbol{T}_l - \boldsymbol{S}_l)W(\hat{u}_l) &= \sum_{j=j_1}^{j_2} \Delta\omega_j N_{j,k}(\hat{u}_l)(\boldsymbol{P}_j - \boldsymbol{T}_l) \\ 2\alpha_j \Delta\,\omega_j &= \sum_{l=0}^{p} N_{j,k}(\hat{u}_l)\boldsymbol{k}_l(\boldsymbol{P}_j - \boldsymbol{T}_l) \\ l &= 0, 1, \cdots, p \end{aligned}\right\} \tag{17}$$

Therefore, Eq.(18) can be derived.

$$\Delta\omega_j = \frac{1}{2\alpha_j}\sum_{t=0}^{p} N_{j,k}(\hat{u}_t)(\boldsymbol{P}_j - \boldsymbol{T}_t)^{\mathrm{T}}\boldsymbol{k}_t \tag{18}$$

Setting $\boldsymbol{B}_j = \frac{1}{\alpha_j}\left[N_{j,k}(\hat{u}_0)(\boldsymbol{P}_j - \boldsymbol{T}_0)^{\mathrm{T}} \cdots N_{j,k}(\hat{u}_p) \cdot (\boldsymbol{P}_j - \boldsymbol{T}_p)^{\mathrm{T}}\right]$ and $\boldsymbol{K} = [\boldsymbol{k}_0 \quad \cdots \quad \boldsymbol{k}_t \quad \cdots \quad \boldsymbol{k}_p]^{\mathrm{T}}_{2p\times 1}$, it follows that

$$\Delta\omega_j = \frac{1}{2}\boldsymbol{B}_j\boldsymbol{K} \qquad j_1 \leqslant j \leqslant j_2 \tag{19}$$

Accordingly, Eq.(17) can be rewritten as

$$(\boldsymbol{T}_l - \boldsymbol{S}_l)W(\hat{u}_l) = \frac{1}{2}\sum_{j=j_1}^{j_2}\sum_{t=0}^{p}\frac{N_{j,k}(\hat{u}_l)N_{j,k}(\hat{u}_t)}{\alpha_j}\cdot (\boldsymbol{P}_j - \boldsymbol{T}_l)(\boldsymbol{P}_j - \boldsymbol{T}_t)^{\mathrm{T}}\boldsymbol{k}_t \tag{20}$$

Write $\boldsymbol{A}^l_{jt} = \frac{N_{j,k}(\hat{u}_l)N_{j,k}(\hat{u}_t)}{\alpha_j}(\boldsymbol{P}_j - \boldsymbol{T}_l)^{\mathrm{T}}(\boldsymbol{P}_j - \boldsymbol{T}_t)$ and $\boldsymbol{A}^l_j = [\boldsymbol{A}^l_{j0} \quad \cdots \quad \boldsymbol{A}^l_{jt} \quad \cdots \quad \boldsymbol{A}^l_{jp}]_{2\times 2p}$ for $0 \leqslant t \leqslant p$, $0 \leqslant l \leqslant p$, then Eq.(20) takes the form as

$$(\boldsymbol{T}_l - \boldsymbol{S}_l)W(\hat{u}_l) = \frac{1}{2}\sum_{j=j_1}^{j_2}\boldsymbol{A}^l_j\boldsymbol{K} \qquad 0 \leqslant l \leqslant p \tag{21}$$

Set

$$\boldsymbol{A} = \begin{bmatrix}\boldsymbol{A}^0\\ \vdots \\ \boldsymbol{A}^l \\ \vdots \\ \boldsymbol{A}^p\end{bmatrix}_{2p\times 2p}, \quad \boldsymbol{C} = \begin{bmatrix}W(\hat{u}_0)(\boldsymbol{T}_0 - \boldsymbol{S}_0)\\ \vdots \\ W(\hat{u}_l)(\boldsymbol{T}_l - \boldsymbol{S}_l) \\ \vdots \\ W(\hat{u}_p)(\boldsymbol{T}_p - \boldsymbol{S}_p)\end{bmatrix}_{2p\times 1}$$

where $\boldsymbol{A}^l = \sum_{j=j_1}^{j_2}\boldsymbol{A}^l_j$, $0 \leqslant l \leqslant p$, then we can write Eq.(21) in unified matrix form.

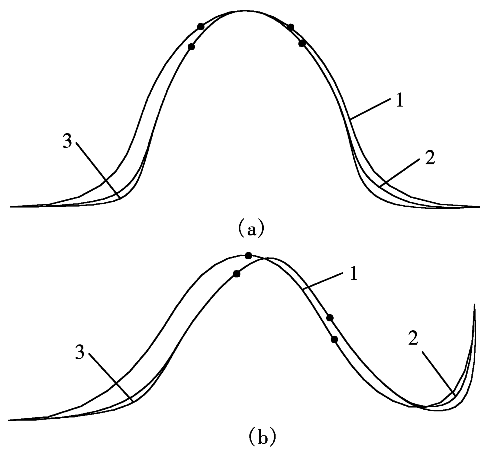
$$\boldsymbol{C} = \frac{1}{2}\boldsymbol{A}\boldsymbol{K} \tag{22}$$

As a result, denoting $\boldsymbol{B} = \begin{bmatrix}\boldsymbol{B}_{j_1}\\ \vdots \\ \boldsymbol{B}_{j_2}\end{bmatrix}$, the explicit solution for multi-target point constraint is governed by the following equation derived from Eq.(19) and Eq.(22):

$$\Delta\boldsymbol{\omega} = \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{C} \tag{23}$$

Fig.6 are examples of multi-target point constraint. Weights ω_i ($i = 1, 2, \cdots, 5$) are changed.

Considering the local support property of B-spline base function, altering a weight ω_i only affects curve segments defined over interval $[u_i, u_{i+k+1}]$. An NURBS curve with degree k and $n+1$ control points has $n-k+1$ curve segments. Let $Q = \{q_1, \cdots, q_i, \cdots, q_{n-k+1}\}$ represent the number of positional constraints applied on each curve segment and $\boldsymbol{V} = \{v_1, \cdots, v_i, \cdots, v_{n-k+1}\}$ represent the number of weights that affect each curve segment among the weights to be altered. In general, for planar NURBS curve we can get a unique solution for a single target point constraint by changing two weights. Therefore, if there appears to be $2q_i > v_i$, no solution can be obtained. Fortunately, in practice we can increase the stiffness of the curve by using a knot insertion algorithm until each interval only contains only one positional constraint.



1 —Original curve; 2 —Modified curve, $\alpha_1 = 0.45$, $\alpha_2 = 0.02$, $\alpha_3 = 0.06$, $\alpha_4 = 0.02$, $\alpha_5 = 0.45$; 3 —Modified curve, $\alpha_1 = 0.01$, $\alpha_2 = 0.01$, $\alpha_3 = 0.96$, $\alpha_4 = 0.01$, $\alpha_5 = \mathbf{0.01}$

Fig.6 Examples of multi-target point constraint

4 Conclusion

This paper presents a new method for shape modification of an NURBS curve by minimizing variation of weights by a least square method. Explicit solutions for single target point constraint and multi-target points constraint are derived to compute new weights. With the introduction of free parameters, not only can the geometric constraints be satisfied, but also the shape of the modified curve can be controlled by designers. Practical examples show the proposed method is acceptable in CAD systems.

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基于权因子变动和约束优化的 NURBS 曲线形状修改

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摘要: 提出了一种通过约束优化改变控制顶点相应的权因子, 进行 NURBS 曲线形状修改的新方法. 运用该方法可使得修改后的 NURBS 曲线满足给定的几何约束, 如单点约束和多点约束. 同时引入了一些自由参数,可以在不破坏几何约束的条件下能进一步改变 NURBS 曲线的形状,而且能使修改后的曲线形状更自然. 由于推导出了明确的公式来计算修改后曲线新的权因子, 因而该方法简单且易于编写程序. 实例表明该方法适用于 CAD 软件系统.

关键词: 非均匀有理 B 样条; 形状修改; 约束优化; 凸包

中图分类号: TP391