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Row-action SAGE algorithm for PET image reconstruction

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Abstract: The one-block version of ordered subsets (OS) techniques is used to accelerate the convergent rate of the space-alternating generalized expectation-maximization (SAGE) algorithm. The new row-action SAGE (RA-SAGE) algorithm processes projections in sequentially orthogonal order which reduce the dependency among the projections and speeds up the convergences. Additionally, the over-relaxation parameter in the direction defined by the RA-SAGE algorithm is also applied to obtain fast convergence to a globally maximum likelihood (ML) solution. In experiments, the RA-SAGE algorithm and the classical SAGE algorithm are compared in the application to positron emission tomography (PET) image reconstruction. Simulation results show that RA-SAGE has better performance than SAGE in both convergence and image quality.

Key words: positron emission tomography; space-alternating; generalized expectation-maximization; row-action; maximum likelihood

Positron emission tomography (PET) images are of great importance to clinical medical diagnoses for their advantages in mapping the changes of radionuclide injected in the tissue of human body. In the past decade, PET reconstruction has benefited greatly from the introduction of statistical reconstruction methods. Unlike the relatively rigid deterministically based method, such as the filtered back-projection (FBP), statistical methods can be applied without modification to data with missing or low signal-to-noise ratios (SNRs) projections. This makes the statistical reconstruction methods well suited for emission problems.

The maximum likelihood expectation-maximization (ML-EM) method for PET image reconstruction was first proposed in Ref. [1], also independently in Ref. [2]. The ML-EM solves the maximum likelihood (ML) estimation by using the expectation-maximization (EM) method, which is based on the notation of a set of "complete" data. However, due to the typical limits in fidelity of data, ML estimation is usually unstable. And the slow convergence of EM has become the greatest disadvantage.

Therefore, various techniques concerning fast convergent rate have been studied, and substantial improvements have been achieved by several methods such as SAGE[2,3] and the ordered subsets (OS)[4]. The SAGE algorithm is representative, thus speeding up the convergent rate by updating parameters using a sequence of cleverly chosen "hidden" data spaces instead of the entire large "complete" data space. Moreover, better choices of these "hidden" data spaces with considerably less Fisher information may yield even more accelerations for the algorithm. Unlike the SAGE algorithm, the OS method takes a strategy that updates parameters grouped on a sequence of ordered subsets (or blocks) of projections. It provides the OS method with high performance, simple implementation and low computational cost. Generally, OS can be applied to any algorithms which involve a sum over projections. However, Ref. [5] noted that it may not be globally convergent in the general inconsistent case without considering OS balance. In this case, they have successfully imposed the OS balance by introducing the relaxation parameter in one-block version of OS and resulted in the so-called row-action maximum likelihood algorithm (RAMLA).

The purpose of this study is to accelerate the SAGE algorithm by using the relaxed row-action method, and to yield a new fast version of SAGE: RA-SAGE. Our speed-up tactic is mainly to perform the reconstruction by accessing projections in a special order. For a more convergent rate, an orthonormal processing order has been considered to reduce the dependency among the projections[6]. In addition, the relaxation parameter is also used to exert an appropriate update level for iteration of reconstruction to keep the convergence to the globally ML solution.

1 Method

In PET image reconstruction, we can assume that

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the photon counts y_i collected on the i-th line of response (LOR) has the Poisson distribution as follows:

$$y_i \sim \text{Poisson}\left(\sum_{j=1}^{N_p} a_{ij}x_j + r_i\right) \tag{1}$$

where r_i is the Poisson background noise mean, a_{ij} denotes the probability that a photon emitted from pixel j results in a coincidence at the i-th LOR, $i=1, 2, \cdots, N_d$ (N_d is the number of LOR), $j=1, 2, \cdots, N_p$ (N_p is the number of pixel). The log-likelihood for this problem is given by[2]

$$\log f(\boldsymbol{y}; \boldsymbol{x}) = \sum_{i=1}^{N_d}[-\hat{y}_i(\boldsymbol{x}) + b_i \log \hat{y}_i(\boldsymbol{x})] \tag{2}$$

$$\hat{y}_i(\boldsymbol{x}) = \sum_{j=1}^{N_p} a_{ij}x_j + r_i \tag{3}$$

Then, starting with a strictly positive vector $\boldsymbol{x}^{(0)}$, we have the EM formula as follows[1]:

$$x_j^{(k+1)} = \frac{x_j^{(k)}}{\sum_{i=1}^{N_d} a_{ij}} \sum_{i=1}^{N_d} \frac{a_{ij}y_i}{\langle \boldsymbol{A}_i^{\mathrm{T}}, \boldsymbol{x}\rangle} \qquad j=1, 2, \cdots, N_p \tag{4}$$

$$\langle \boldsymbol{A}_i^{\mathrm{T}}, \boldsymbol{x}\rangle = \sum_{j=1}^{N_p} a_{ij}x_j \tag{5}$$

where $\langle \cdot \rangle$ represents the inner product, $\boldsymbol{A}_i$ is the i-th row of projection matrix $\boldsymbol{A}$, $x_j^{(k)}$ is the expected value of pixel j at iteration k.

The RAMLA method is a faster alternative to the ML-EM algorithm that updates projections in a special order[5]. Its update for a given pixel j can be expressed as

$$x_j^{(k,t)} = x_j^{(k,t-1)} + \lambda_k x_j^{(k,t-1)} \cdot \sum_{l \in S_t} a_{lj}\left(\frac{y_l}{\langle \boldsymbol{A}_l^{\mathrm{T}}, \boldsymbol{x}^{(k,t-1)}\rangle} - 1\right)$$
$$t=1, 2, \cdots, p; \; j=1, 2, \cdots, N_p \tag{6}$$

where $\boldsymbol{x}^{(k,0)} = \boldsymbol{x}^{k-1}$, $\bigcup_{t=1}^{p} S_t = [1, 2, \cdots, m]$, $c_t = \max \sum_{l \in S_t} a_{lj}$, and $0 < \lambda_k c_t \leqslant 1$. If set $\lambda_k = 1/c_t$, we can obtain the "one-block" version OS-EM method as follows:

$$x_j^{(k,t)} = \frac{x_j^{(k,t-1)}}{\sum_{l \in S_t} a_{lj}} \sum_{l \in S_t} a_{lj} \frac{y_l}{\langle \boldsymbol{A}_l^{\mathrm{T}}, \boldsymbol{x}^{(k,t-1)}\rangle} \tag{7}$$

Though the RAMLA is the special case of OS-EM, it imposes the OS balance with relaxing techniques and guarantees the convergence to the globally maximum likelihood solution. Mathematical proof can be found in Ref. [5]. In this paper, we adapt the row-action method into the SAGE algorithm and then yield the new RA-SAGE algorithm. Like OS-EM, the update is mainly performed on a group of single projection spaces. Therefore, the RA-SAGE algorithm is as follows:

Initializing: $\hat{y}_i = \sum_{j=1}^{N_p} a_{ij}x_j^{(0)} + r_i$, $i=1, 2, \cdots, N_d$, $\boldsymbol{x}^{(0)}$ is a positive start image.

for $k=1, 2, \cdots$ {(loop until convergence)

for $j=1, 2, \cdots, N_p$ {(update pixel sequentially)

$$\text{E-step: } e = y_i/\hat{y}_i - 1, \; e_j = \sum_{i=1}^{N_d} a_{ij}e \tag{8}$$

$$\text{M-step: } \begin{cases} x_j^{(k+1)} = x_j^{(k)} + \lambda_k x_j^{(k)} \sum_{i \in S_t} e_j/a_j^* \\ a_j^* = \sum_{i=1}^{N_d} a_{ij}; \; t=1, 2, \cdots, p \\ x_q^{(k+1)} = x_q^{(k)} \qquad q \neq j \end{cases} \tag{9}$$

$$\hat{y}_i = \hat{y}_i + (x_j^{(k+1)} - x_j^{(k)})a_{ij} \qquad \forall i, \; a_{ij} \neq 0, \; i \in S_t \tag{10}$$

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Compared with the SAGE algorithm[2,3], we can find that the RA-SAGE keeps the sequential update for pixels but performs projection update (10) on the grouped single projection spaces S_t, $t=1, 2, \cdots, p$. Moreover, the M-step (9) has been optimized by using the relaxation parameters λ_k. These not only provide the RA-SAGE with a faster convergent rate than that of SAGE but also guarantee the update towards a truly globally maximum likelihood solution. In addition, the access order for grouped projections can influence the convergence of our algorithm[6]. Therefore, to improve the convergent rate, special processing order can be considered in our RA-SAGE algorithm. A recommended method can be found in Ref. [6] where projections are sorted in such a way as to reserve the smallest dependency among the projections. In this paper, the angularly orthogonal order has been taken into account (see the following experiments for more details).

2 Experimental Results

In the simulations, we have compared the reconstructions of RA-SAGE with those of SAGE. Fig.1 shows the phantom that is used for testing the structural recovery of the reconstruction. This phantom is similar to the phantom proposed by Eiichi and Hiroyuki[7]. It consists of an elliptic uniform disc, a circular hot area, a cold area and a sharp spot. The circle with a diameter of 90 mm for cold area, a diameter of 120 mm for hot area, and relative activities of the elements are shown in Fig.1. The total mean photon counts in a series of projection data is 10^6, which is also studied by including 5% uniform Poisson background noise, representing the range of random coincidences in PET scans. The sinogram has 128 radial bins and 180 angular views. The size of a pixel is 6 mm × 6 mm and the size of the image matrix is 384 mm × 384 mm. As a result, the reconstructed images are 64 × 64 pixel matrices.

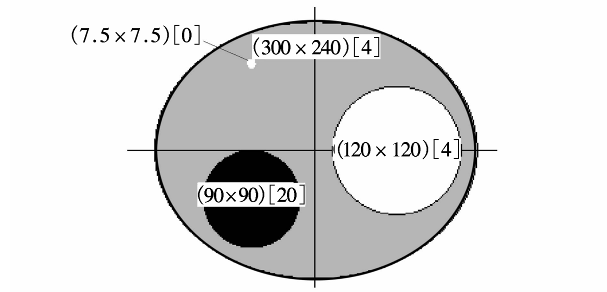

Fig.1 A simulated phantom

Fig. 2 shows the reconstructed images obtained by SAGE and RA-SAGE. In the RA-SAGE algorithm, we modify the number of projections and access order of the subsets described in Ref. [6]. Fig.3 shows an example of the access order for 4 subsets. The choice of the relaxation parameter depends on a number of factors[7], such as the medical purpose of the reconstruction, the geometry of data collection, and the characteristics of the projection data, etc. One can freely choose an appropriate parameter which fits the

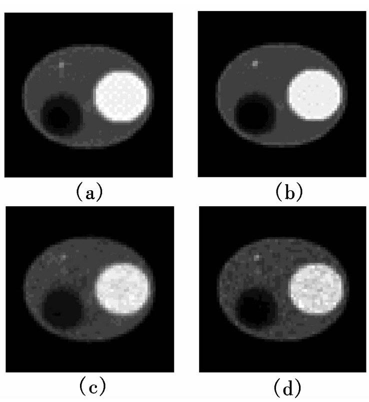

Fig.2 Reconstructed images. (a) SAGE reconstruction using noise free projections; (b) RA-SAGE reconstruction using noise free projections; (c) SAGE reconstruction using projections with Poisson background noise; (d) RA-SAGE reconstruction using projections with Poisson background noise (The iteration number is 20 in SAGE, and 5 in RA-SAGE).

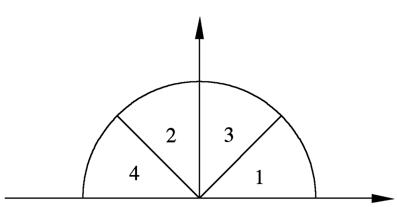

Fig.3 Access order for 4 subsets (the order is shown in number)

specific experimental data. In our experiments, the relaxation parameter λ_k takes the value 1.08.

Fig.4 shows the log-likelihood versus iteration number for SAGE and RA-SAGE algorithms. Clearly we can see that the log-likelihood of RA-SAGE increases more remarkably than the one of SAGE. To evaluate the quality of the final reconstructions, the mean absolute error (MAE) and Chi-square error (CSE) are both employed in our experiments. The MAE is defined as

$$\varepsilon_{\mathrm{MAE}} = \frac{1}{N_{\mathrm{p}}} \sum_{j=1}^{N_{\mathrm{p}}} \left| x_j^{\mathrm{rec}} - x_j^{\mathrm{org}} \right| \tag{11}$$

where x_j^{org} denotes the value of pixel j of the original activity image and x_j^{rec} denotes the value of pixel j of a reconstructed image. It measures the average discrepancy between a reconstructed image and the original activity image. CSE is defined as

$$\varepsilon_{\mathrm{CSE}} = \sum_{i=1}^{N_{\mathrm{d}}} \left[y_i \log(y_i / \hat{y}_i^{(k)}) - (y_i - \hat{y}_i^{(k)}) \right] \tag{12}$$

$$\hat{y}_i^{(k)} = \sum_{j=1}^{n} a_{ij} x_j^{(k)} + r_i \tag{13}$$

which measures the discrepancy between calculated projections and the original projections. Fig. 5 shows the MAE changes versus the iteration number of different reconstruction methods, while Fig. 6 shows the CSE changes via the iteration number.

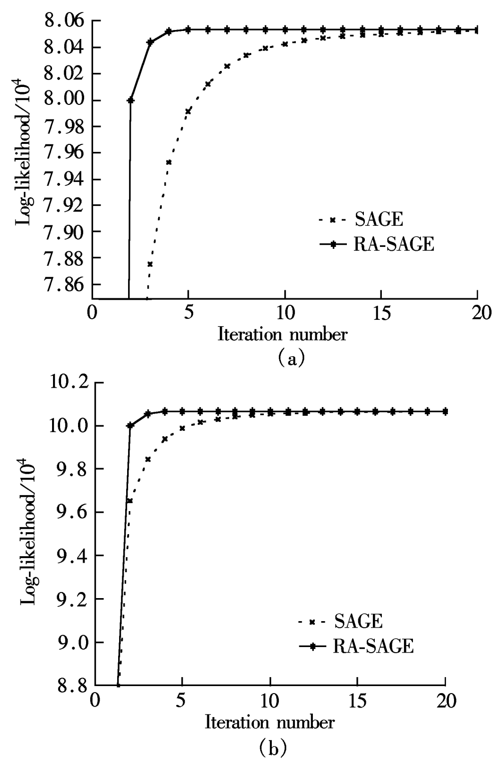

Fig.4 Comparison of log-likelihood increase $\log f(\boldsymbol{y}; \boldsymbol{x}^{(k)}) - \log f(\boldsymbol{y}; \boldsymbol{x}^{(0)})$ versus iteration number k for SAGE and RA-SAGE algorithms. (a) Noise free projections; (b) Projections with Poisson background noise

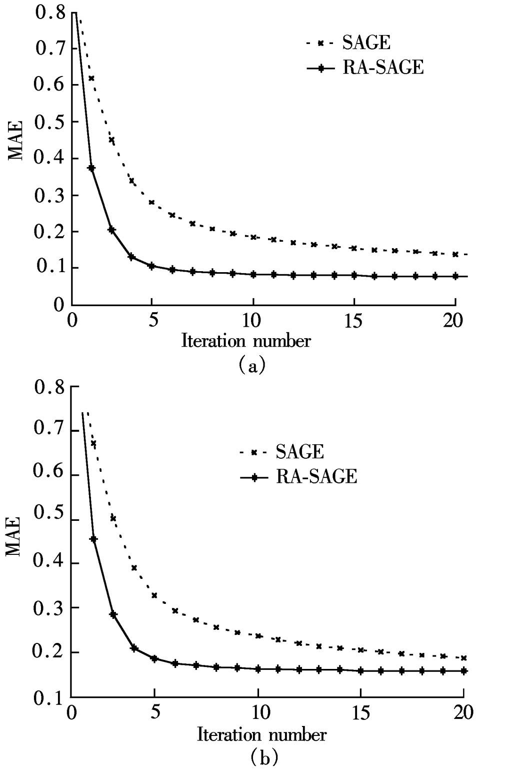

Fig.5 Mean absolute error of reconstructed images. (a) Noise free projections; (b) Projections with Poisson background noise

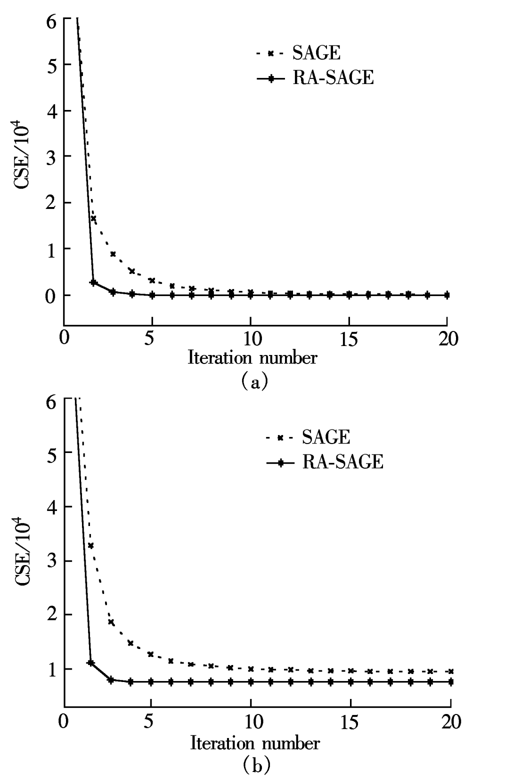

Fig.6 Chi-square error of calculated projection data for various algorithms. (a) Noise free projections, (b) Projections with Poisson background noise

Clearly, our RA-SAGE algorithm yields the smaller MAE and CSE than the SAGE algorithm, which means that our algorithm can offer better image quality than the SAGE algorithm.

3 Conclusion

In this paper, an effective method for accelerating the convergence in PET image reconstruction has been proposed. The resultant RA-SAGE algorithm is mainly based on the SAGE algorithm in which the row-action method has been successfully incorporated. Experimental studies have clearly shown that the RA-SAGE method yields reconstructions with faster convergences than the SAGE algorithm, which also indicates that our algorithm is helpful for improving the image quality of PET reconstructions.

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基于行处理的 SAGE 算法在 PET 图像重建中的应用

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摘要: 运用基于行处理(RA)的“单块”投影子集法改进了空间交替广义期望最大(SAGE)算法的收敛性. 新的 RA-SAGE 算法以正交单投影序列的方式对投影数据进行处理, 以减少投影间的相关性, 达到加速收敛的效果. 此外, 在迭代搜索同时, 新算法结合了超松弛变量, 使其能快速接近全局最大似然解. 实验中, 运用 RA-SAGE 与 SAGE 对正电子发射断层(PET)进行了重建. 结果表明,RA-SAGE 收敛性能比 SAGE 优越, 且重建图像质量较高.

关键词: 正电子发射断层; 空间交替; 广义期望最大; 行处理; 最大似然

中图分类号: R817