Journal of Southeast University (English Edition) Vol. 20 No. 4 Dec. 2004 ISSN 1003—7985

Limit cycle problem for quadratic differential system $\dot{x} = -y + lx^2 + mxy, \dot{y} = x(1 + ax + by)$

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Abstract: The maximal number of limit cycles for a particular type Ⅲ system $\dot{x} = -y + lx^2 + mxy$, $\dot{y} = x(1 + ax + by)$ is studied and some errors that appeared in the paper by Suo Mingxia and Yue Xiting (*Annals of Differential Equations*, 2003, 19(3):397 −401) are corrected. By translating the system to be considered into the Liénard type and by using some related properties, we obtain several theorems with suitable conditions coefficients of the system, under which we prove that the system has at most two limit cycles. The conclusions improve the results given in Suo and Yue's paper mentioned above.

Key words: quadratic differential system; limit cycle; weak focus

1 Particular Type Ⅲ Quadratic System with $d = n = 0$

According to Ye's classification, the quadratic differential system can be considered in the form[1]:

$$\left.\begin{aligned}\dot{x} &= -y + dx + lx^2 + mxy + ny^2\\ \dot{y} &= x(1 + ax + by)\end{aligned}\right\} \tag{1}$$

which is called the general type Ⅲ system if $b \neq 0$. We now consider the particular case with $d = n = 0$ and $m \neq 0$. Then by scaling we can take $m = 1$, the system to be considered is

$$\left.\begin{aligned}\dot{x} &= -y + lx^2 + xy\\ \dot{y} &= x(1 + ax + by)\end{aligned}\right\} \tag{2}$$

For system (2), $O(0, 0)$ has a weak focus with the order at least one.

Ref. [2] made a conjecture: if a quadratic system has a weak focus of order α and there are β limit cycles surrounding this focus, then $\alpha + \beta \leqslant 3$. For the cases of $\alpha = 3$ and $\alpha = 2$, the conjecture has been proved[3]. In this paper, we will prove that the quadratic system (2) has at most two limit cycles (LCs) surrounding its first order weak focus.

The focal quantities of system (2) at the origin $O(0, 0)$ which are discussed in Ref. [4], are expressed by

$$\bar{V}_7 = -a^3 l(1 - 15l^2)$$

$$\bar{V}_5 = (1 - 5a)\frac{a^2 - 1 + 2a}{a^2}$$

$$\bar{V}_3 = l - a(b + 2l)$$

$$\bar{V}_1 = 0$$

respectively. We think that $\bar{V}_5$ and $\bar{V}_7$ are not correct. Because for system (1), when $d = 0$, the focal quantities are[1]

$$W_3 = m(l + n) - a(b + 2l)$$

$$W_5 = ma(5a - m)[(l + n)^2(n + b) - a^2(b + 2l + n)]$$

$$W_7 = ma^2[2a^2 + n(l + 2n)] \cdot [(l + n)^2(n + b) - a^2(b + 2l + n)]$$

where W_3, W_5 and W_7 use the notations of Ref. [1], which are the same as $\bar{V}_3$, $\bar{V}_5$ and $\bar{V}_7$ in Ref. [4], respectively.

For system (2) with $m = 1$, $n = 0$, we obtain

$$\left.\begin{aligned}W_3 &= l - a(b + 2l)\\ W_5 &= a(5a - 1)[bl^2 - a^2(b + 2l)]\\ W_7 &= 2a^4[bl^2 - a^2(b + 2l)]\end{aligned}\right\} \tag{3}$$

Even if $W_3 = 0$, that is $l = a(b + 2l)$ or $b = \frac{1 - 2a}{a}l$, then

$$W_5 = l(1 - 5a)[a^2 + l^2(2a - 1)]$$

$$W_7 = -2a^3 l[a^2 + l^2(2a - 1)]$$

Thus the expressions of $\bar{V}_5$ and $\bar{V}_7$ in Ref. [4] cannot be obtained in any way. For example, if $a = 2/5$, $b = 1$, $l = 2$, then $W_3 = 0$, $W_5 = 32/25$ and $W_7 = 2^9/5^5$, but $\bar{V}_3 = 0$, $\bar{V}_5 = 1/4$ and $\bar{V}_7 = 59 \cdot 2^4/5^3$.

2 Reducing to the Liénard Type System

As in Ref. [4], system (2) can be reduced to the Liénard type system by the following transformations.

First, let $\xi = -y + lx^2 + xy$, $x = x$, then we have

$$\dot{x} = \xi$$

$$\dot{\xi} = -(x + (a - 1)x^2) - (lb - a)x^3 - \frac{-(b + 2l)x + (l + b)x^2}{1 - x}\xi - \frac{1}{1 - x}\xi^2$$

for which we see that there are some errors that appeared in Ref. [4].

Received 2004-06-14.

Foundation item: The National Natural Science Foundation of China (No. 19871041).

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Secondly, let $u = \xi/(1-x)$, $x = x$, $\mathrm{d}\tau = (1-x)\mathrm{d}t$, then the system is converted into

$$\left.\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}\tau} &= u \\ \frac{\mathrm{d}u}{\mathrm{d}\tau} &= -x(g(x)+f(x)u)\end{aligned}\right\} \tag{4}$$

where

$$g(x) = \frac{1+(a-1)x+(bl-a)x^2}{(1-x)^2}$$

$$f(x) = \frac{-(b+2l)+(l+b)x}{(1-x)^2}$$

Eq.(4) is obviously a Liénard type system.

3 Some Lemmas

The lemma in Ref. [4] points out that the limit cycle around the origin cannot intersect the curve $g(x)+f(x)u=0$ under the following additional conditions: $l>1/\sqrt{15}$, $0<a<1/5$, $l-a(b+2l)>0$, $lb>a$, $\bar{V}_3>(b+3l)(bl-a)$.

In fact, by the properties of limit cycles for quadratic systems, such additional conditions are not needed for the conclusion. We now prove the following lemma.

Lemma 1 For any value of a, b, l, any LC surrounding the origin of system (2) does not intersect the curve $g(x)+f(x)u=0$.

Proof Any closed orbit surrounding the origin of system (2) is a convex closed curve (see Ref. [1]), as shown in Fig.1, for which the highest and lowest points are located on y-axis (a branch of the horizontal isocline line) and the left most and right most points are located on the branch through the origin of $-y+lx^2+xy=0$. This closed orbit cannot intersect another branch of the horizontal isocline: $1+ax+by=0$.

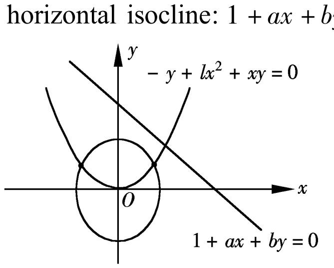

Fig.1 Closed orbit in (x, y) plane

The transformations $x=x$, $\xi=-y+lx^2+xy$ keep $x=0$ invariant and convert the other branch $1+ax+by=0$ into $g(x)+f(x)u=0$ for (4) and the vertical isocline $-y+lx^2+xy=0$ into $\xi=0$ which corresponds to $u=0$ for system (4).

Without loss of generality, we assume that $l \geqslant 0$ (otherwise, changing (y, t) into $(-y, -t)$ to get the case). For system (2), we have $\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{x-1=0} = l > 0$.

So $x=1$ is a straight line without contact (in the case $l=0$, the system has an invariant line and the origin has a weak focus, then has no limit cycle), the closed orbit of system (2) does not intersect $x=1$, thus it remains in the part of $x<1$, in which the above transformations are 1-1 correspondence.

The closed orbit L around the origin of system (4) is shown in Fig.2, for which the highest and lowest points are still on $x=0$ (u-axis) and the furthest left and right points are on the new x-axis (corresponding to $-y+lx^2+xy=0$). Thus L cannot intersect the other branch $g(x)+f(x)u=0$. Fig.2 also shows if a trajecture intersects with the branch $g(x)+f(x)u=0$, then it turns to the upright and cannot be surrounded by $O(0,0)$.

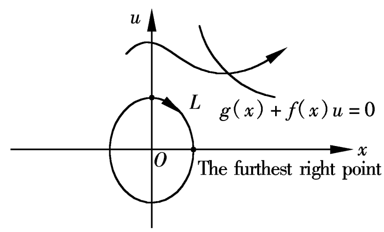

Fig.2 Closed orbit in (x, u) plane

Lemma 1 is proved.

For the curve $u=-\frac{g(x)}{f(x)}$, a branch of horizontal isocline of system (4),

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{f^2(x)}[-g'(x)f(x)+f'(x)g(x)] = \frac{1}{f^2(x)}[a(b+2l)-l+2(b+2l)(bl-a)x-(b+l)(bl-a)x^2] = \frac{1}{f^2(x)}H(x)$$

where $H(x) = -(b+l)(bl-a)x^2+2(b+2l)(bl-a)x+a(b+2l)-l$.

We have the following conclusion.

Lemma 2 If $bl-a>0$, $l \leqslant \frac{2a+1-4b^2+\sqrt{(2a+1)^2+b^2}}{8b}$, then $H(x) \leqslant 0$.

Proof The discriminant of $H(x)$ is

$$\begin{aligned}\Delta_1 &= 4(b+2l)^2(bl-a)^2 + \\ &\quad 4(b+l)(bl-a)[a(b+2l)-l] = \\ &\quad 4l(bl-a)[4bl^2+(4b^2-2a-1)l+ \\ &\quad b^3-ab-b]\end{aligned}$$

Let

$$\varphi(l) = 4bl^2+(4b^2-2a-1)l+b^3-ab-b \tag{5}$$

which is considered as a quadratic polynomial of l with the discriminant

$$\Delta_2 = (4b^2-2a-1)^2-16b(b^3-ab-b) = (2a+1)^2+8b^2>0$$

The roots of $\varphi(l)=0$ are

$$l_1 = \frac{2a + 1 - 4b^2 + \sqrt{(2a+1)^2 + 8b^2}}{8b}$$

$$l_2 = \frac{2a + 1 - 4b^2 - \sqrt{(2a+1)^2 + 8b^2}}{8b}$$

Now $bl - a > 0$, $l > 0$, then $b > 0$. Thus $-(b+l)\cdot(bl-a) < 0$. If

$$l \leqslant \frac{2a + 1 - 4b^2 + \sqrt{(2a+1)^2 + 8b^2}}{8b}$$

then $\varphi(l) \leqslant 0$, and $\Delta_1 \leqslant 0$. Thus we obtain that $H(x) \leqslant 0$.

Remark 1 For the condition $l \leqslant \frac{2a + 1 - 4b^2 + \sqrt{(2a+1)^2 + 8b^2}}{8b}$ to be satisfied, we need the condition

$$2a + 1 - 4b^2 + \sqrt{(2a+1)^2 + 8b^2} > 0 \qquad b > 0$$

which is equivalent to that of $b^2 < a + 1$.

4 Main Conclusions

Theorem 1 For $bl - a > 0$, $0 < a < 1/5$, system (2) possesses two limit cycles surrounding the origin under certain additional conditions.

Proof It is proved that if $W_3 W_5 > 0$, $W_5 W_7 > 0$, there is no limit cycles around O (see Ref. [5]). We now discuss the case of $W_3 W_5 < 0$, $W_5 W_7 < 0$.

Firstly, let $W_3 = 0$, then $l = a(b + 2l)$ and

$$W_5 W_7 = 2a^5 l(5a-1)(bl-a)^2 < 0$$

which is equivalent to $a(5a-1) < 0$ or $0 < a < 1/5$.

Starting from system (2) with O being a third order weak focus, we have $a = 1/5$ and $l - (b+2l)/5 = 0$, i. e. $b = 3l$. Then $W_3 = W_5 = 0$ and $W_7 = 2a^4(bl - a) \neq 0$. For $bl - a > 0$ (or $bl - a < 0$), O is unstable (or stable). To change its stability, let $a < 1/5$ with $1/5 - a$ small enough and $W_3 = 0$ still satisfied, then $W_5 < 0$ (or > 0) and there is an unstable (or a stable) LC L_1 created from $O(0, 0)$.

Again, let b decrease (or increase) a little such that $W_3 = l - a(b + 2l) > 0$ (or < 0) and L_1 still exists, then the stability of $O(0, 0)$ changes again to get a second LC $L_2 \subset L_1$ with L_2 stable (or unstable).

The conclusion of theorem 1 is obtained.

In the following, we are going to prove that there are at most two limit cycles.

Theorem 2 Under the conditions: $bl - a > 0$, $0 < a < \frac{1}{5}$, $0 < l < \frac{2a + 1 - 4b^2 + \sqrt{(2a+1)^2 + 8b^2}}{8b}$, system (2) has at most two LCs surrounding the origin.

Proof Let $L_2 \subset L_1$ be any two-neighboured LCs around $O(0, 0)$ of system (4), which are both obviously clockwise. For $i = 1, 2$, we have

$$\oint_{L_i} \operatorname{div}(4)\,\mathrm{d}t = \oint_{L_i} -xf(x)\,\mathrm{d}t = \oint_{L_i} \frac{f(x)}{g(x) + f(x)u}\mathrm{d}u \tag{6}$$

Lemma 1 implies that $g(x) + f(x)u \neq 0$ on the existential domain of the limit cycles around $O(0, 0)$, and then $g(x) + f(x)u > 0$ (because $g(0) + f(0)u = 1 > 0$). Therefore, we know that $\frac{f(x)}{g(x) + f(x)u}$ is a continuously differentiable function for $(x, u) \in D$, where D is an annular region bounded by L_1 and L_2, see Fig.3.

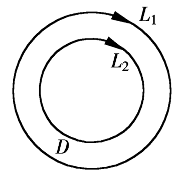

Fig.3 Annular region bounded by L_1 and L_2

Changing the orientation of L_1 and applying the Green's formula to region D, we obtain

$$\left(-\oint_{L_1} + \oint_{L_2}\right)\frac{f(x)}{g(x) + f(x)u}\mathrm{d}u = \iint_D \frac{\partial}{\partial x}\left(\frac{f(x)}{g(x) + f(x)u}\right)\mathrm{d}x\mathrm{d}u = \iint_D \frac{H(x)}{(g(x) + f(x)u)^2}\mathrm{d}x\mathrm{d}u \tag{7}$$

The right hand side of Eq.(7) is negative due to lemma 2. For the two LCs obtained in theorem 1, L_1 is unstable and L_2 stable, then the left hand side of Eq.(7) is also negative and there is no contradiction. But if there are three LCs around $O(0, 0)$, then we can choose the two-neighboured LCs with outside one stable (at least inner stable) and inside one unstable (at least outer unstable). For this case, we get that the left hand side of Eq.(7) is greater than or equal to zero, and obtain a contradiction.

Theorem 2 is proved.

Remark 2 For the case $bl - a < 0$, by starting from a stable weak focus of order three, then the stabilities of the LCs obtained in theorem 1 change to the opposite. Thus, to satisfy the proof in theorem 2, it requires the conclusion that $H(x) \geqslant 0$, and it is sufficient that $\varphi(l) \geqslant 0$ in Eq.(5), and so that $\Delta_1 \leqslant 0$.

We look at the figures of $\varphi(l)$, a quadratic polynomial of l with at least one negative zero point l_2, then we can get the following conclusions.

Lemma 3 $\varphi(l) \geqslant 0$ if one of the following conditions is satisfied:

1) $b > 0$, $bl - a < 0$, $b^2 \geqslant a + 1$;

2) $b>0$, $bl-a<0$, $b^2<a+1$ and $l \geqslant \frac{2a+1-4b^2+\sqrt{(2a+1)^2+8b^2}}{8b}$;

3) $b<0$ (then $bl-a<0$), $b^2<a+1$ and $l \leqslant \frac{2a+1-4b^2+\sqrt{(2a+1)^2+8b^2}}{8b}$.

4) $b<0$, $b^2>a+1$ and $\frac{2a+1-4b^2+\sqrt{(2a+1)^2+8b^2}}{8b} \leqslant l \leqslant \frac{2a+1-4b^2-\sqrt{(2a+1)^2+8b^2}}{8b}$.

Proof From remark 1, it is enough to see Fig.4.

By the above conclusions, we have the following theorem.

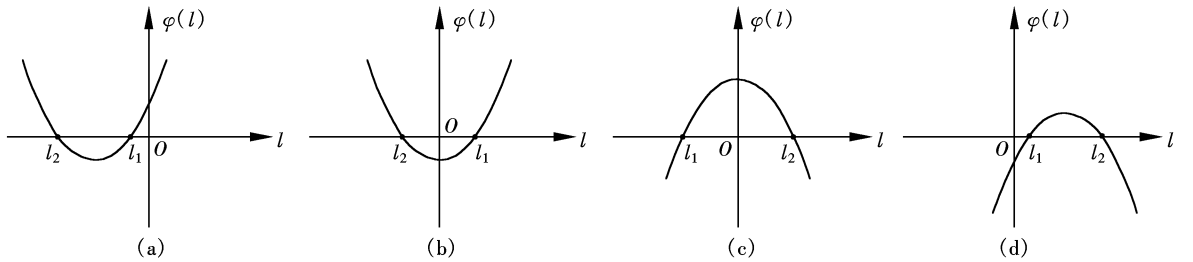

Fig.4 $\varphi(l)$ graphs

Theorem 3 System (2) has at most two LCs around $O(0, 0)$ if $0<a<1/5$ and one of the conditions in lemma 3 is satisfied.

Remark 3 The conditions in theorem 2 are simpler than those in theorem of Ref. [4] for the case $bl-a>0$. Theorem 3 discusses the case $bl-a<0$ which is not considered in Ref. [4].

We studied the limit cycle problem for system (2) in more detail. The related results will be written in a separate article.

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二次微分系统 $\dot{x}=-y+lx^2+mxy$, $\dot{y}=x(1+ax+by)$ 的极限环问题

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摘要: 研究了特殊Ⅲ类二次微分系统 $\dot{x}=-y+lx^2+mxy$, $\dot{y}=x(1+ax+by)$ 的极限环的最大个数问题. 纠正了索明霞和岳锡亭的文章(微分方程年刊, 2003,19(3):397 − 401)中的一些错误. 通过将所研究的系统化为Liénard型系统并利用其相关性质给出了几个定理, 在某些条件下证明了系统最多存在 2 个极限环, 改进了上述一文中的结果.

关键词: 二次微分系统; 极限环; 细焦点

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