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Priority approach based on quadratic programming model to fuzzy preference relation

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Abstract: We investigate the decision-making problem with a finite set of alternatives, in which the decision information takes the form of a fuzzy preference relation. We develop a simple and practical approach to obtaining the priority vector of a fuzzy preference relation. The prominent characteristic of the developed approach is that the priority vector can generally be obtained by a simple formula, which is derived from a quadratic programming model. We utilize the consistency ratio to check the consistency of fuzzy preference relation. If the fuzzy preference relation is of unacceptable consistency, then we can return it to the decision maker to reconsider structuring a new fuzzy preference relation until the fuzzy preference relation with acceptable consistency is obtained. We finally illustrate the priority approach by two numerical examples. The numerical results show that the developed approach is straightforward, effective, and can easily be performed on a computer.

Key words: decision making; fuzzy preference relation; quadratic programming; priority

In the process of decision making with a finite set of alternatives, a decision maker generally needs to compare these alternatives with a single criterion, and constructs a preference relation. Fuzzy preference relation is one of the most common preference relations[1,2]. How to derive a rational priority vector for a fuzzy preference relation is an interesting and important issue[2]. In this paper, we shall develop a simple and practical priority approach to fuzzy preference relation.

1 A Priority Approach to Fuzzy Preference Relation

Consider a decision-making problem. Let $X = \{x_1, x_2, \dots, x_n\}$ be a finite set of alternatives. The decision maker gives his/her preference information on X by means of a fuzzy preference relation $\boldsymbol{B} = (b_{ij})_{n\times n}$, where $b_{ij} \in (0, 1)$, $b_{ij} + b_{ji} = 1$, $b_{ii} = 0.5$, $i, j = 1, 2, \dots, n$, and b_{ij} denotes the preference degree or intensity of the alternative x_i over x_j.

Definition Let $\boldsymbol{B} = (b_{ij})_{n\times n}$ be a fuzzy preference relation. If the following multiplicative transitivity[3] is satisfied:

$$b_{ik}b_{kj}b_{ji} = b_{ij}b_{jk}b_{ki} \qquad \text{for all } i, j, k$$

then $\boldsymbol{B}$ is called the multiplicative consistent fuzzy preference relation.

Let $\boldsymbol{v} = \{v_1, v_2, \dots, v_n\}^{\mathrm{T}}$ be the priority vector of the fuzzy preference relation $\boldsymbol{B} = (b_{ij})_{n\times n}$, where $v_i > 0$, $i = 1, 2, \dots, n$, $\sum_{i=1}^{n} v_i = 1$.

If $\boldsymbol{B} = (b_{ij})_{n\times n}$ is a multiplicative consistent fuzzy preference relation, then such a preference relation is given by[4]

$$b_{ij} = \frac{v_i}{v_i + v_j} \qquad i, j = 1, 2, \dots, n \tag{1}$$

From Eq. (1), we have the following system of equations:

$$b_{i1}(v_i + v_1) + b_{i2}(v_i + v_2) + \dots + b_{in}(v_i + v_n) = nv_i \qquad i = 1, 2, \dots, n \tag{2}$$

i. e.,

$$b_{i1}v_1 + b_{i2}v_2 + \dots + b_{i,i-1}v_{i-1} + \left(b_{ii} + \sum_{j=1}^{n} b_{ij}\right)v_i + b_{i,\,i+1}v_{i+1} + \dots + b_{in}v_n = nv_i \qquad i = 1, 2, \dots, n \tag{3}$$

System (3) can be represented in a matrix form[5]:

$$(\boldsymbol{B} + \mathrm{diag}(\boldsymbol{Be}))\boldsymbol{v} = n\boldsymbol{v} \tag{4}$$

where $\boldsymbol{e}$ denotes a uniform vector of the n-th order, diag($\boldsymbol{Be}$) of totals in each row of matrix $\boldsymbol{B}$ corresponds to sums added to diagonal items in system (3), and

$$\boldsymbol{e}^{\mathrm{T}}\boldsymbol{v} = 1 \tag{5}$$

By Eqs. (4) and (5), we have

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$$\boldsymbol{v} = \left\{\left(\sum_{i=1}^{n} \frac{b_{i1}}{b_{1i}}\right)^{-1}, \left(\sum_{i=1}^{n} \frac{b_{i2}}{b_{2i}}\right)^{-1}, \ldots, \left(\sum_{i=1}^{n} \frac{b_{in}}{b_{ni}}\right)^{-1}\right\}^{\mathrm{T}} \quad (6)$$

However, people's judgements depend on personal psychological aspects, such as experience, learning, situation, state of mind, and so forth[6], hence, the consistency condition is rarely satisfied. As a result, in the general case, Eq. (4) does not hold. Here, we introduce the deviation vector $\boldsymbol{\varepsilon} = \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}^{\mathrm{T}}$, i. e. ,

$$\boldsymbol{\varepsilon} = (\boldsymbol{B} + \mathrm{diag}(\boldsymbol{Be}))\boldsymbol{v} - n\boldsymbol{v} = (\boldsymbol{B} + \mathrm{diag}(\boldsymbol{Be}) - n\boldsymbol{I})\boldsymbol{v} \quad (7)$$

where $\boldsymbol{I}$ is the unit matrix.

By Eq. (7), we construct the deviation function

$$f(\boldsymbol{v}) = \boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{\varepsilon} \quad (8)$$

That is,

$$f(\boldsymbol{v}) = \boldsymbol{v}^{\mathrm{T}}(\boldsymbol{B} + \mathrm{diag}(\boldsymbol{Be}) - n\boldsymbol{I})^{\mathrm{T}} \cdot (\boldsymbol{B} + \mathrm{diag}(\boldsymbol{Be}) - n\boldsymbol{I})\boldsymbol{v} \quad (9)$$

Obviously, a reasonable priority vector $\boldsymbol{v}$ should be determined so as to minimize $f(\boldsymbol{v})$, that is, the following quadratic programming model can be established.

$$\min f(\boldsymbol{v}) = \boldsymbol{v}^{\mathrm{T}}(\boldsymbol{B} + \mathrm{diag}(\boldsymbol{Be}) - n\boldsymbol{I})^{\mathrm{T}} \cdot (\boldsymbol{B} + \mathrm{diag}(\boldsymbol{Be}) - n\boldsymbol{I})\boldsymbol{v} \quad (10)$$

If $\boldsymbol{B}$ is a multiplicative consistent fuzzy preference relation, then we can obtain the priority vector $\boldsymbol{v} = \{v_1, v_2, \ldots, v_n\}^{\mathrm{T}}$ by using Eq. (6).

If $\boldsymbol{B}$ is not a multiplicative consistent fuzzy preference relation, then, following the technique of Wang and Xu[7] and solving Eq. (10), we get the optimal solution as follows:

$$\boldsymbol{v}^{*} = \frac{((\boldsymbol{B} + \mathrm{diag}(\boldsymbol{Be}) - n\boldsymbol{I})^{\mathrm{T}}(\boldsymbol{B} + \mathrm{diag}(\boldsymbol{Be}) - n\boldsymbol{I}))^{-1}\boldsymbol{e}}{\boldsymbol{e}^{\mathrm{T}}((\boldsymbol{B} + \mathrm{diag}(\boldsymbol{Be}) - n\boldsymbol{I})^{\mathrm{T}}(\boldsymbol{B} + \mathrm{diag}(\boldsymbol{Be}) - n\boldsymbol{I}))^{-1}\boldsymbol{e}} \quad (11)$$

If $\boldsymbol{v}^{*} \leqslant \boldsymbol{0}$, then we can solve Eq. (10) by using a quadratic programming method, and get the attribute weights.

To check the consistency of the fuzzy preference relation $\boldsymbol{B} = (b_{ij})_{n \times n}$, Xu and Da[8] gave a formula as follows:

$$\left.\begin{aligned} I_{\mathrm{c}}(\boldsymbol{B}) &= \frac{1}{n(n-1)} \sum_{1 \leqslant i < j \leqslant n} \left(\frac{b_{ij}}{b_{ji}} \frac{v_j}{v_i} + \frac{b_{ji}}{b_{ij}} \frac{v_i}{v_j} - 2\right) \\ R_{\mathrm{c}}(\boldsymbol{B}) &= \frac{I_{\mathrm{c}}(\boldsymbol{B})}{I_{\mathrm{m}}} \end{aligned}\right\} \quad (12)$$

where $\boldsymbol{v} = \{v_1, v_2, \ldots, v_n\}^{\mathrm{T}}$ is the priority vector of $\boldsymbol{B}$, $I_{\mathrm{c}}(\boldsymbol{B})$ and $R_{\mathrm{c}}(\boldsymbol{B})$ are the consistency index and the consistency ratio of $\boldsymbol{B}$, respectively, and I_{m} is the mean consistency index given by Saaty[1], which is listed in Tab. 1.

Tab. 1 The mean consistency index

n	I_{m}	n	I_{m}
1	0	9	1.46
2	0	10	1.49
3	0.52	11	1.52
4	0.89	12	1.54
5	1.12	13	1.56
6	1.26	14	1.58
7	1.36	15	1.59
8	1.41		

If $R_{\mathrm{c}}(\boldsymbol{B}) < 0.1$, then the fuzzy preference relation $\boldsymbol{B}$ is of acceptable consistency; otherwise, we can return $\boldsymbol{B}$ to the decision maker to reconsider structuring a new fuzzy preference relation until the fuzzy preference relation with acceptable consistency is obtained (We can also utilize the approach in Ref. [8] to improve the consistency of $\boldsymbol{B}$).

2 Numerical Examples

In this section, two numerical examples are given to illustrate the developed priority procedure.

Example 1 For a decision-making problem, there are four decision alternatives x_1, x_2, x_3 and x_4. The decision maker provides his/her preferences over these four decision alternatives, and gives a fuzzy preference relation as follows:

$$\boldsymbol{B}_1 = \begin{bmatrix} 0.5 & 0.8 & 0.4 & 0.6 \\ 0.2 & 0.5 & 0.3 & 0.7 \\ 0.6 & 0.7 & 0.5 & 0.6 \\ 0.4 & 0.3 & 0.4 & 0.5 \end{bmatrix}$$

By Eq. (11), we get the following priority vector of $\boldsymbol{B}_1$:

$$\boldsymbol{v}^{*} = \{0.2960,\ 0.1642,\ 0.3707,\ 0.1690\}^{\mathrm{T}}$$

By Eq. (12), we obtain the consistency ratio of $\boldsymbol{B}_1$ as $R_{\mathrm{c}}(\boldsymbol{B}_1) = 0.1471 > 0.1$. Since the fuzzy preference relation $\boldsymbol{B}_1$ is of unacceptable consistency, then we return $\boldsymbol{B}_1$ to the decision maker to reconsider structuring a new fuzzy preference relation. Suppose that the restructured fuzzy preference relation is

$$\boldsymbol{B}_1' = \begin{bmatrix} 0.5 & 0.8 & 0.6 & 0.8 \\ 0.2 & 0.5 & 0.3 & 0.7 \\ 0.4 & 0.7 & 0.5 & 0.6 \\ 0.2 & 0.3 & 0.4 & 0.5 \end{bmatrix}$$

By Eq. (11), we get the following priority vector of $\boldsymbol{B}_1'$:

$$\boldsymbol{v}^{*} = \{0.4658, 0.1422, 0.2748, 0.1172\}^{\mathrm{T}}$$

By Eq. (12), we obtain the consistency ratio of $\boldsymbol{B}_1'$ as $R_{\mathrm{c}}(\boldsymbol{B}_1') = 0.0648 < 0.1$. Thus, the fuzzy preference relation $\boldsymbol{B}_1'$ is of acceptable consistency. Using $\boldsymbol{v}^{*}$, we get the ranking of these four alternatives: $x_1 \succ x_3 \succ x_2 \succ x_4$. Therefore, the most desirable alternative is x_1.

Example 2 For a decision-making problem, there are five decision alternatives x_1, x_2, x_3, x_4 and x_5. The decision maker provides his/her preferences over these five decision alternatives, and gives a fuzzy preference relation as follows:

$$\boldsymbol{B}_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{2} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{2} & \frac{2}{3} & \frac{4}{5} & \frac{4}{5} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{2} & \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{5} & \frac{1}{3} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{5} & \frac{1}{3} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Since $\boldsymbol{B}_2$ is a multiplicative consistent fuzzy preference relation, $R_c(\boldsymbol{B}_2) = 0$, then by Eq. (6), we get the following priority vector of $\boldsymbol{B}_2$:

$$\boldsymbol{v}^* = \{0.2, 0.4, 0.2, 0.1, 0.1\}^{\mathrm{T}}$$

and thus, the ranking of these five alternatives is

$$x_2 > x_1 \sim x_3 > x_4 \sim x_5$$

Therefore, the most desirable alternative is x_2.

3 Conclusion

In this paper, we have developed a simple priority procedure for a fuzzy preference relation. By using this procedure, the priority vector of a fuzzy preference relation can generally be obtained through calculating a simple formula based on a quadratic programming model. The theoretical analysis and the numerical results show that the developed approach is straightforward, effective, and can easily be performed on a computer.

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一种基于二次规划模型的模糊偏好关系排序法

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摘要: 研究了决策信息以模糊偏好关系给出的有限方案决策问题, 提出了一种简洁且实用的模糊偏好关系排序方法. 该方法首先建立一个二次规划模型, 然后基于该模型推导出求解模糊偏好关系排序向量的一个简洁公式. 基于获得的排序向量, 利用一致性比例对模糊偏好关系进行一致性检验. 对于一致性较差的模糊偏好关系, 则需反馈给决策者重新进行判断, 直至得到一个一致性可接受的模糊偏好关系为止. 最后, 利用 2 个算例对该方法进行分析和说明, 数值结果表明该方法简洁、有效, 且易于在计算机上操作.

关键词: 决策; 模糊偏好关系; 二次规划; 排序

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